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Waves on glaciers

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This paper is an attempt at a mathematical synopsis of the theory of wave motions on glaciers. These comprise surface waves (analogous to water waves) and seasonal waves (more like compression waves). Surface waves have been often treated and are well understood, but seasonal waves, while observed, do not seem to have attracted any theoretical explanation. Additionally, the spectacular phenomenon of glacier surges, while apparently a dynamic phenomenon, has not been satisfactorily explained.

The present thesis is that the two wave motions (and probably also surging, though a discussion of this is not developed here) can both be derived from a rational theory based on conservation laws of mass and momentum, provided that the basal kinematic boundary condition involving boundary slip is taken to have a certain reasonable form. It is the opinion of this author that the form of this 'sliding law' is the crux of the difference between seasonal and surface waves, and that a further understanding of these motions must be based on a more satisfactory analysis of basal sliding.

Since ice is here treated in the context of a slow, shallow, non-Newtonian fluid flow, the theory that emerges is that of non-Newtonian viscous shallow-water theory; rather than balance inertia terms with gravity in the momentum equation, we balance the shear-stress gradient. The resulting set of equations is, in essence, a first-order nonlinear hyperbolic (kinematic) wave equation, and susceptible to various kinds of analysis. We show how both surface and seasonal waves are naturally described by such a model when the basal boundary condition is appropriately specified. Shocks can naturally occur, and we identify the (small) diffusive parameters that are present, and give the shock structure: in so doing, we gain a useful understanding of the effects of surface slope and longitudinal stress in these waves.

1. Introduction

There are at least two main areas of geophysics in which one is led to consider the deformation of solids from the point of view of fluid mechanics. One is the convection of the earth's mantle (e.g. Turcotte & Oxburgh 1972), which is studied in relation to the theory of plate tectonics; the other is the flow of glaciers and ice sheets. Although the materials in both cases are crystalline solids, each is capable of shearing motion by mechanisms such as dislocation creep (e.g. Stocker & Ashby 1973). Over sufficiently long time scales, it thus seems that the motion of these solids may be adequately described by a rheological equation of state which relates the strain-rate tensor to the stress tensor. In mantle convection, relevant time scales are measured in millions of

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years. In glaciers and ice sheets, the scale of large-scale fluid-like motions is upwards of 10 years, and is as large as 5000 years for the Antarctic ice sheet. Thus it is reasonable to consider ice as a fluid when seeking to explain the various phenomena that are observed to occur in large ice masses. In so doing, we equivalently consider ice as a homogeneous continuum, thus ignoring short-time-scale phenomena such as crevasse formation, iceberg calving, etc., which may be analysed in the context of an elastic solid (e.g. Weertman 1977).

This procedure is certainly a valid one to adopt in discussing the particular phenomena with which we shall be principally concerned. These are *surface waves, seasonal waves* and *surges*. In terms of quantitative explanations, these three types of glacial motion have had varying degrees of success. Surface waves are undulations of the glacial surface profile which travel downglacier at speeds (typically) of three to four times the surface speed, which itself is typically of the order of 100 m per year (there is of course a substantial amount of variation in this figure). These waves are completely analogous to ordinary flood waves on rivers, and indeed are briefly discussed in Whitham's (1974) book in the context of his kinematic wave theory. The waves propagate as a consequence of the equation of mass conservation, and many analyses using this idea have been given, stemming from work by Finsterwalder (1907), and more recently by Weertman (1958), Nye (1960) and Lliboutry (1965, 1971). In its essentials, this theory solves the problem of surface waves, although some of the mathematical detail remains to be discussed.

Seasonal waves are much less well understood. Additionally, they do not seem to be comparable to any other kind of fluid wave motion, and a satisfactory analytical explanation of their mode of propagation would therefore be welcome. Seasonal waves manifest themselves as fluctuations in the surface *velocity* which propagate downglacier at speeds in the range 20–150 times the surface speed: they can thus travel at speeds of the order of magnitude of 20 km per year. Although the fluctuations in speed are large, there is no apparent corresponding depth perturbation; thus the glacier surface remains relatively static during the propagation of these velocity waves. As such, these bear some resemblance to acoustic waves, and indeed they are referred to as 'Druckwellen' (pressure waves) in early work by Deeley & Parr (1914). One of the objects of the present study is to examine whether the compressive effects of longitudinal strain-rate gradients are relevant in this context. Another early reference is the work of Blümcke & Finsterwalder (1905); recent work on Nisqually glacier by Hodge (1974) documents the passage of seasonal waves in 1969.

The third and most interesting phenomenon exhibited by glaciers is that of surging. A surge is a large-scale relaxation oscillation of a large portion of a glacial ice mass. The main characteristics of the motion have been summarized by Meier & Post (1969): briefly, a 'surge-type' glacier exhibits a 'quiescent' phase lasting from 10 to 100 years, typically 20–30 years, during which the ice mass increases owing to accumulation of ice on its upper portions (the 'accumulation area'). Consequently the thickness of the ice increases in a 'reservoir' area, until it reaches an apparently critical depth, when a surge is suddenly initiated. In surging mode, the ice in the reservoir area moves very rapidly, and there is a substantial displacement of ice downstream into a 'receiving' area. In some cases, this can lead to rapid advances of the glacier, and velocities up to about 10 km per year have been recorded. Evidence for the periodicity of surges comes from examination of terminal and medial moraines, and it therefore seems that surging is a self-regulating mechanism due to internal glacier flow dynamics, rather than being the result of any periodic external forcing due, for example, to fluctuation in the climate. From a mathematical point of view, this is a satisfactory state of affairs, since it suggests that a theoretical explanation should be found from a study of the equations of glacier flow alone; to date, this theory has not appeared. This is not to say that physical mechanisms, computer predictions, etc., have not been published: there are indeed a variety of different theories available (Robin 1955, 1969; Robin & Weertman 1973; Budd 1975, Campbell & Rasmussen 1969, Clarke, Nitsan & Paterson 1977; Cary, Clarke & Peltier, 1979; Yuen & Schubert 1979): the mechanisms discussed include a multi-valued basal sliding law, thermal runaway due to shear heating instability, instability of basal water flow, and so on. What seems to be lacking is a single mathematical framework in which to examine all these theories. This is not trivial, in view of the wealth of dynamic possibilities available, but it is one aim of the present paper to provide just such a framework. This is based on the concept of a rational model of glacier flow, proposed and developed elsewhere (Fowler & Larson 1978; Fowler 1979*a*); the model is rational in the sense that it purports to give a precise description of an entity that *mathematically* closely resembles a glacier; for example, the geometry is not arbitrarily restricted by fixed boundaries, but the top surface is allowed to remain free. This turns out to have a crucial effect on the uniqueness of the flow (Fowler & Larson 1980*a*), and is obviously necessary for a study of surface wave motions. In principle, we should hope that the model would have built into it an explanation of all the phenomena described above and, if so, it will then emerge what the physical role is of the parameters which are present in the model. This has already been done for surface waves (Fowler & Larson 1980*b*), and briefly for seasonal waves (Fowler 1979*a*): the purpose of the present paper is to give a unified quantitative explanation of both these types of wave, and in particular to analyse the diffusional structure of shock waves which occur naturally in the diffusionless limit. In particular (because the equations are essentially viscous), there are two diffusion parameters, and we are able to identify these and estimate their relative importance.

The corresponding study of surges is reserved for a future paper: we shall only make some further comments in the conclusions. Nevertheless, we have included a brief discussion of their properties here since we feel a proper theoretical understanding of the dynamic processes involved (for example the propagation of high-velocity regions *upstream*) can be firmly based on a knowledge of how both surface and seasonal waves propagate. For example, the quiescent phase of a surge occupies a time comparable to that which a kinematic wave takes to traverse the glacier length, while the surging phase has the time scale of passage of a seasonal wave. This is not fortuitous, and a surge might correspondingly be viewed in terms of a switching mechanism between 'kinematic' and 'seasonal' states. The techniques for analysing this switching are those promoted here.

It is also worth mentioning that periodic surges of the Antarctic ice sheet have been suggested as a constituent of ice ages (Bowen 1980; Hollin 1980; Aharon, Chappel & Compston 1980). Obviously, such a surge has never been directly observed, and a realistic explanation of surging could thus provide a sensible criterion for evaluating the possibilities of such an event occurring.

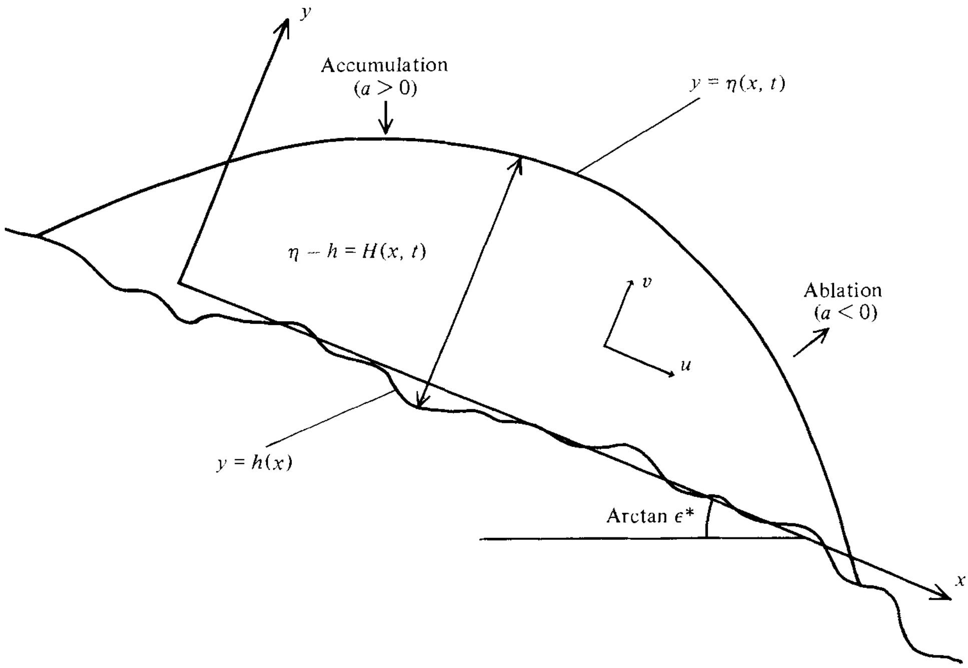

FIGURE 1. Geometry of glacier flow.

2. Mathematical model

A detailed model for the flow of glacial ice masses has been derived and presented elsewhere (Fowler & Larson 1978; Fowler 1979*a*), and it is not our intention to go through this whole process here. Rather, we shall outline some important physical processes which arise in such a derivation.

In what follows, we consider a two-dimensional ice flow as shown in figure 1. The two-dimensionality reflects the fact that the phenomena that interest us here are essentially of this nature; moreover, this would be an adequate description in wide-valley glaciers, where the main flow is unaffected by the valley walls. We take Cartesian axes (x, y) as shown in the diagram, with x pointing down the line of mean bedrock slope: the corresponding velocity components are (u, v). The top surface is denoted by $y = \eta(x, t)$, the bedrock is $y = h(x)$, assumed known; the surface $\eta(x, t)$ is a free boundary, which is to be determined as part of the solution. For the moment, we assume that h is arbitrary; later we shall take $h = 0$ as a simplifying, but *inessential* assumption. We define the glacier depth (perpendicular to the line of mean bedrock slope) as

$$H(x,t) \equiv \eta(x,t) - h(x). \tag{2.1}$$

A glacier is a long but importantly *finite* flow. A typical length might be 10 km. The flow is maintained by the accumulation of fallen snow in an accumulation area at the glacier *head*, which is subsequently packed into ice. This manifests itself as a source term on the top surface. Equivalently (since the climate is warmer downglacier), the ice melts or *ablates* there (in an ablation zone), which is why the glacier terminates at its *snout*. The line dividing accumulation and ablation areas is called the firn or equilibrium line. A useful introduction to the physics of glaciers is the book by Paterson (1969).

The constitutive law that describes the flow behaviour of ice is generally assumed to be of the non-Newtonian form

$$e = A(S)f(\tau), \tag{2.2}$$

where the second invariants e and τ are defined by

$$2e^2 = e_{ij}e_{ij}, \quad 2\tau^2 = \tau_{ij}\tau_{ij}, \tag{2.3}$$

the summation convention being understood, and S is a state variable. For cold ice below its melting point, S is the temperature; for temperate ice at its melting point, the temperature is effectively constant (depending only weakly on the pressure), and the relevant state variable is the moisture content of the ice (Lliboutry 1976). The form usually chosen for (2.2) is called Glen's law, after laboratory work by Glen (1955):

$$e = A(S)\tau^n. \tag{2.4}$$

Here n is an exponent whose measured values lie close to 3. In fact (Budd & Radok 1971), laboratory and field experiments indicate that n increases with τ, so that a better form for $f(\tau)$ would be an expression such as

$$f(\tau) = \sinh^n\left(\frac{\tau}{\tau_0}\right). \tag{2.5}$$

However, neither (2.4) nor (2.5) has the property that $f'(0) \neq 0$, which is a necessary condition that the viscosity be finite at zero stress. Other laws having this property can be constructed, for example a polynomial law (Colbeck & Evans 1973), or the Ellis model (Bird 1976):

$$f(\tau) = a_1\tau + a_2\tau^m. \tag{2.6}$$

Actually, (2.5) is probably better, and has the added advantage that it is invertible, which simplifies computation.

Glen's law is sufficiently accurate over the range of stresses encountered in glaciers ($\tau \lesssim 1$ bar), and we shall adopt it here for the reason that it is the most widely accepted form. Also, it has not been suggested, and we do not suspect, that the precise form of $f(\tau)$ has more than a quantitative bearing on the behaviour of solutions.

This is not true of the state dependence $A(S)$, which introduces (in cold ice) coupling of the flow and temperature fields. This situation also arises in the study of the flow of the earth's mantle (Yuen & Schubert 1977), polymers and plastics (Pearson 1977, 1978) and chemical reactors (Gavalas 1968), where it is known that such phenomena as multiple steady states and thermal runaway (Gruntfest 1963) can occur; these phenomena have been suggested as a possible mechanism of surging by Clarke *et al.* (1977) and Yuen & Schubert (1979), although this last contention is controversial (Fowler 1980; Fowler & Larson 1980*a*, *c*). Although consideration of the state variable renders the problem less tractable, it has not been suggested that its inclusion is relevant to a study of wave motions, and we do not suggest so here. Because of this, we shall simply set $A(S) =$ constant, and suppose that this simplification will do no more than quantitatively affect the results.

At the bedrock, a glacier is capable of boundary slip if the basal ice is temperate, which is quite realistic even in cold (polar) regions owing to viscous heating of the ice. This is because an extremely thin film of melt water separates the ice from the bedrock due to pressure melting or regelation (Nye 1967). This is the phenomenon whereby increased pressure upstream of a bedrock protrusion lowers the local melting point, whereas decreased pressure downstream increases it: the resulting (small) temperature difference induces a heat flux through ice and rock which melts a film of water (typically of thickness $\sim 10^{-6}$ m), which then flows to the downstream side of the protrusion where it refreezes. By this mechanism, the shear stress on the ice at the bed is zero, and the ice can slide over the bed: glaciological observations of this basal slip are commonplace (e.g. Kamb & LaChapelle 1964). In this case the basal shear stress of the main ice flow is taken up at the bed by the pressure variation (i.e., although the shear stress *at* the bedrock is zero, that 'seen' by the flow away from the bed is a mean stress due to pressure fluctuations exerted by a smoothed bed profile). In order to determine the size of this shear stress, it is necessary to solve the local bedrock flow and *match* it (via matched asymptotic expansions) into the 'outer' flow. The matching condition then provides a functional relation between the basal stress τ_b and the basal velocity u_b (Fowler 1979*b*), which we write as

$$\tau_b = f(u_b). \tag{2.7}$$

If the bedrock is sufficiently rough, then (2.7) gives $u_b \simeq 0$ for $\tau_b \sim O(1)$ (both u_b and τ_b are supposed scaled and dimensionless). This is effectively the normal no-slip condition (Richardson 1973). If we measure the 'slope' of the bedrock roughness by a parameter $\nu^* \lesssim 1$ (which may be defined as the maximum magnitude of the slope of the rough bedrock relative to the average profile), then Fowler (1981) showed that $u_b \sim O(\sigma^*/\nu^{*n+1})$, where σ^* is the ratio of bedrock roughness amplitude to the glacier depth, and n is the exponent in Glen's law. Thus if $\sigma^* \simeq 10^{-2}$, sliding is non-trivial if $\nu^* \sim \frac{1}{2}$, substantial if $\nu^* \sim \frac{1}{3}$, and dominant if $\nu^* < \frac{1}{5}$. It is therefore quite realistic to consider glaciers whose dominant motion is due to basal sliding: indeed, this is the case for the Nisqually glacier (Hodge 1974). The proper scaling of the equations of motion in this last case is given in appendix A.

The form of the sliding law has been studied extensively, principally by Weertman (1957), Lliboutry (1968), Nye (1969, 1970), Kamb (1970), and others more recently. Two recent and contrasting reviews are those by Weertman (1979) and Lliboutry (1979). In the absence of ice–bedrock separation (to be discussed later) and with very small scale microrelief, one form of this law is

$$\tau_b = R u_b^{1/n} \tag{2.8}$$

(Fowler 1981), where n is the exponent in Glen's flow law, and R is a roughness parameter. Other forms have been mooted, but the important property is that τ_b increases with u_b, as would be expected on physical grounds. The complicating effect of water at the bed has led Lliboutry and others to develop theories to encompass this possibility: we return to this in due course; Bindschadler (1979) has shown that in such cases a completely different sliding law may be preferable.

To close this section, we recall from Fowler & Larson (1978) the principal equations with which we shall be concerned: these are now cast in dimensionless, scaled $O(1)$) form. The kinematic wave equation

$$H_t + Q_x = s_x(x,t) \tag{2.9}$$

governs the (dimensionless) depth $H(x,t)$. Here the flux

$$Q = \int_h^\eta u\,dy, \tag{2.10}$$

and the flux function $s(x,t)$ represents the source on $y = \eta$ due to accumulation and ablation. The task of determining Q will be addressed in § 3. To do so, we require the equations of momentum balance. These can be written in the dimensionless form

$$\tau_{2y} = -1 + \mu\eta_x + \delta^2 p_x - \delta^2 \tau_{1x}, \tag{2.11}$$

$$p_y = \tau_{2x} - \tau_{1y}, \tag{2.12}$$

where we neglect inertial terms, as the Reynolds number is typically $\sim 10^{-13}$. In (2.11) and (2.12), τ_1 and τ_2 are the longitudinal and transverse stress-deviator components, and p is the pressure (minus its hydrostatic component). The parameter μ represents effects of surface slope variations, and δ represents the shallowness of the flow. We may think of $\mu \sim 10^{-1}$, $\delta \sim 10^{-2}$, typically. The term -1 in (2.11) represents longitudinal (downglacier) gravity acceleration, and indicates that the balance we have chosen is that between shear-stress gradient and this driving acceleration.

The appropriate stress conditions on $y = \eta$ then turn out to be

$$\left.\begin{aligned} \tau_2 + \delta^2(p-\tau_1)\,\eta_x &= 0, \\ \tau_1 + p + \tau_2\eta_x &= 0, \end{aligned}\right\} \tag{2.13}$$

and lastly the constitutive flow law may be written in the form

$$\left.\begin{aligned} \tau_1 &= 2u_x \bar{e}^{-(n-1)/n}, \\ \tau_2 &= (u_y + \delta^2 v_x)\,\bar{e}^{-(n-1)/n}, \\ \bar{e} &= [(u_y+\delta^2 v_x)^2 + 4\delta^2 u_x^2]^{\frac{1}{2}}. \end{aligned}\right\} \tag{2.14}$$

3. Scaling considerations

Our immediate objective is to derive an expression for the flux Q by finding the x velocity component. How we do this depends on what we assume about u_b, and what levels of accuracy we retain. If we put $\delta = 0$, we find, from (2.11), (2.13) and (2.14), that

$$\tau_2 = (\eta - y)\,[1-\mu\eta_x] = u_y^{1/n} \tag{3.1}$$

provided $\tau_2 > 0$, i.e. if $\eta_x < 1/\mu$. It follows that

$$u = u_b + \frac{(1-\mu\eta_x)^n}{n+1}\,[H^{n+1} - (\eta - y)^{n+1}], \tag{3.2}$$

and thus

$$Q = Hu_b + \frac{H^{n+2}}{n+2}\,[1-\mu(H_x+h_x)]^n. \tag{3.3}$$

Now the term $\delta^2\tau_{1x}$ in (2.11) is a longitudinal compression term, and thus in considering seasonal waves, where u may change rapidly with x, this term may act in a diffusive manner for seasonal shock waves. We therefore wish to examine under what circumstances it can be important.

If we balance $\delta^2\tau_{1x}$ with τ_{2y} in the y momentum equation, then it turns out that such a shock region leads to a messy elliptic problem, which, we consider, somewhat obscures the fundamentally one-dimensional nature of such a region.

What we really want to do is to seize on this *one-dimensionality* of seasonal waves, that is, that the glacier behaves almost like a rigid 'elastic' body during the passage of these waves (Hodge 1974; Lliboutry & Rayneaud 1981). Apart from simplifying the analysis, this is essentially what is observed. One way to do this is to suppose that the basal sliding velocity is formally much larger than the deformation due to shearing; this would be the case if the bed roughness were comparatively small (or if cavitation were present, see § 5). In this case (which may accurately represent what happens during a surge), it is appropriate to write the scaled velocity field in the form

$$u = u_{\mathrm{b}}(x,t)+\beta u_1(x,y,t), \quad v = -yu_{\mathrm{b}}'+\beta v_1(x,y,t) \quad (\beta \ll 1), \tag{3.4}$$

where a prime denotes differentiation with respect to x. The parameter β is a measure of the ratio of the velocity difference between surface and base due to shearing within the ice to the basal sliding velocity. It is determined entirely by the physical inputs to the model and the basal sliding law, also regarded as an input (as boundary condition). It is thus an externally controllable parameter, and to consider it small (e.g. $\beta \sim \frac{1}{10}$) is equivalent to supposing that basal sliding is predominant, as for the Nisqually glacier (Hodge 1974).

At this stage (having shown how the bedrock h manifests itself in the flux (3.3)) we put $h = 0$, and we do not consider that this will be an important simplification provided $h' \sim O(1)$ in practice (which rules out ice falls, for example). The scaling of the problem analogous to that in Fowler & Larson (1978) is now slightly different, and is carried out in appendix A. The resulting version of (2.11)–(2.14) is

$$\tau_1 = 2[u_{\mathrm{b}}'+\beta u_{1x}]\,\bar{e}^{1/n-1}, \tag{3.5a}$$

$$\tau_2 = [u_{1y}-\delta\nu u_{\mathrm{b}}''y+\delta^2 v_{1x}]\,\bar{e}^{1/n-1}, \tag{3.5b}$$

$$\bar{e} = [\{u_{1y}-\delta\nu u_{\mathrm{b}}''+\delta^2 v_{1x}\}^2+4\nu^2\{u_{\mathrm{b}}'+\beta u_{1x}\}^2]^{\frac{1}{2}}, \tag{3.5c}$$

$$p_y+\tau_{1y} = \beta\tau_{2x}, \tag{3.5d}$$

$$\tau_{2y} = -1+\mu\eta_x+\nu\delta(p_x-\tau_{1x}), \tag{3.5e}$$

$$\tau_2 = \nu\delta\eta_x(-p+\tau_1), \quad p+\tau_1 = -\beta\tau_2\eta_x \quad \text{on} \quad y=\eta. \tag{3.5f}$$

Here

$$\delta = \nu\beta \tag{3.6}$$

defines ν, and we assume $\nu \lesssim 1$, i.e. $\beta \gtrsim \delta$. If $\beta \ll \delta$, the glacier really is almost rigid, and the scales change again. This may be relevant for surge propagation, but we omit any consideration of it here.

Now suppose that u_{b}, and hence u, changes rapidly through a distance $x \sim \sigma \ll 1$. (Note that σ and ν here are unrelated to the roughness parameters discussed in § 2.) If we suppose that $\nu/\sigma \lesssim 1$, then $\bar{e} \sim 1$ in (3.5c) and so $\nu\delta\tau_{1x}$ in (3.5e) is ~ 1 only if $\nu\delta/\sigma^2 \sim 1$, i.e. if, from (3.6), $\beta\nu^2 = \nu\delta \sim \sigma^2$. By assumption $\sigma \gtrsim \nu$, so $\sigma^2 \gtrsim \nu^2$, thus $\beta \sim \sigma^2/\nu^2 \gtrsim 1$, which is inadmissible by assumption in (3.4). It follows that $\nu\delta\tau_{1x}$ becomes non-negligible in (3.5e) only if $\nu \gg \sigma$, so that

$$\bar{e} \simeq 2\nu|u_{\mathrm{b}}'| \tag{3.7}$$

in this region, and

$$\tau_1 = 2u_{\mathrm{b}}'|2\nu u_{\mathrm{b}}'|^{1/n-1}. \tag{3.8}$$

Thus in the compression zone,

$$\bar{e}^{1/n-1} \sim \left(\frac{\nu}{\sigma}\right)^{1/n-1}, \quad \tau_1 \sim \frac{\nu}{\sigma}\left(\frac{\nu}{\sigma}\right)^{1/n-1}; \tag{3.9}$$

we suppose however that τ_2 remains $O(1)$, as is physically reasonable, thus

$$\nu\delta\tau_{1x} \sim \frac{\nu\delta}{\sigma^2}\left(\frac{\nu}{\sigma}\right)^{1/n-1}, \tag{3.10}$$

which suggests we choose

$$\frac{\nu\delta}{\sigma^2}\left(\frac{\nu}{\sigma}\right)^{1/n-1} = 1, \quad \text{i.e.} \quad \sigma = \delta/\beta^{1/(n+1)} \tag{3.11}$$

as the rescaling factor. In the compression zone, we expect $p \sim \tau_1$. Now

$$\beta\tau_{2x}/\tau_{1y} \sim \beta/\sigma\tau_1 \sim \beta/(\sigma/\nu)^{(n-1)/n} \sim \beta^{2/(n+1)} \ll 1,$$

therefore (3.5d) is approximately (to $O(\beta^{2/(n+1)})$)

$$p_y + \tau_{1y} = 0. \tag{3.12}$$

Equations (3.5f) imply

$$p + \tau_1 = \delta^2\eta_x^2(p - \tau_1) \quad \text{on} \quad y = \eta; \tag{3.13}$$

even if η changes by $O(1)$ in the compression zone, the right-hand side of (3.13) is $O(\delta^2/\sigma^2) \sim \beta^{2/(n+1)}$, hence, to $O(\beta^{2/(n+1)})$,

$$p = -\tau_1; \tag{3.14}$$

thus, in the compression zone, (3.5e) may be written to leading order as

$$\tau_{2y} = -1 + \mu\eta_x - 2\nu\delta\tau_{1x}, \tag{3.15}$$

where τ_1 is a function of x and t given by (3.8). The boundary condition on $y = \eta$ is, from (3.5f),

$$\tau_2 = 2\nu\delta\tau_1\eta_x \quad \text{on} \quad y = \eta, \tag{3.16}$$

and thus (3.15) implies

$$\tau_2 = (\eta - y)[1 - \mu\eta_x + 2\nu\delta\tau_{1x}] + 2\nu\delta\tau_1\eta_x. \tag{3.17}$$

From (3.17) we derive the expression for the basal stress τ_b by putting $y = h = 0$, $\eta - h = H$, to obtain

$$\tau_b = H[1 - \mu H_x] + 2\nu\delta(H\tau_1)_x. \tag{3.18}$$

Using (3.8) we write this in the form

$$\tau_b = H[1 - \mu H_x] + \alpha[Hu_b'|u_b'|^{1/n-1}]_x, \tag{3.19}$$

where the parameter α is defined by

$$\alpha = 2\delta(2\nu)^{1/n}. \tag{3.20}$$

We recall that $u_b' = \partial u_b/\partial x$ in (3.19). The formula (3.19) shows that, by assuming $\beta \ll 1$ in (3.4), we may (via the scaling analysis of the appendix) derive an explicit formula for τ_b, whereas for the elliptic problem which is relevant when $\beta \sim 1$, such an expression is not easily obtainable. A typical estimate for α is given, when $\delta \sim 10^{-2}$, $\beta \sim 10^{-1}$, $n = 3$, by $\alpha \simeq 1{\cdot}2 \times 10^{-2}$; thus we may take $\alpha \ll 1$ and neglect compressive stresses, except in those regions where u (and hence u_b) changes by $O(1)$ in distances $x \ll 1$.

There are two points to note. Firstly, the reasoning above is based on the consideration that $\tau_{2y} \sim \delta\nu\tau_{1x}$; we left $\mu\eta_x$ out of the discussion. This is legitimate if the jump in η is small enough, but in any case $\delta\nu\tau_{1x}$ can only be non-negligible if it is $\gtrsim \tau_{2y}$, and the argument proceeds as before. Secondly, the formula for the basal stress above parallels the arguments of Robin (1967) and Collins (1968), and its relation to their work should be clarified. Their work seeks a correction term to Nye's classical (1952) expression for the basal stress in terms of the depth h and the angle of inclination α of the top surface to the horizontal (not the same α as used here): $\tau_b = \rho g h \sin\alpha$. In essence, the arguments mirror that presented here, except that our whole procedure is based on dimensionless variables, and also the corrective term in (3.19) is computed on the basis that it becomes asymptotically comparable to the other term $H(1-\mu H_x)$, which is just a direct transcription to dimensionless form of Nye's formula (Fowler 1981). Robin's motivation was to seek corrected profiles of ice sheets due to bedrock irregularity. We deliberately exclude bedrock irregularity ($h = 0$), but seek the corrective terms which may act as diffusion effects in dynamic wave propagation. In fact, the only change to (3.19) if $h \neq 0$ is that H_x in the first bracket is replaced by $H_x + h_x$; however, this is only the case if $h_x \lesssim O(1)$, and thus the present analysis is inapplicable to the problem of determining the basal stress when $h_x \gg 1$, which is exactly the case of interest to Robin (1967) and subsequent workers (e.g. Budd & Radok 1971); this latter is still a matter of some controversy (e.g. Hutter 1981). Our approach is complementary to the aims of these authors.

We can now consider the basis for an analytic study of wave motions. In the case that $\beta \ll 1$, so that the motion is predominantly by sliding, then the expression (3.19) gives a leading-order estimate for the basal stress, assumed functionally related to the basal velocity, in (thin) regions where compressive stress is appreciable. Now, although (3.19) is not correctly derived away from such regions, it is asymptotically equivalent (since $\alpha \ll 1$) to the leading-order version of the exact equation (with $\delta = 0$) there. Therefore it is reasonable to suppose that (3.19) will provide a suitable expression for the basal stress in the whole ice domain of interest, at least for the study of the kinds of wave that concern us. We are now in a position to commence this study; before doing so, we summarize the main results of this section.

From the original model of two-dimensional glacier flow presented by Fowler & Larson (1978), we have derived a kinematic wave equation by neglecting any effects of the state variable on the flow law (in (2.2)), it being our contention that such a neglect is not of qualitative relevance to the analysis given here. In considering this equation, we have isolated two particular variants which will (as we show below) suffice to explain the structure of both surface and seasonal waves.

The kinematic wave equation for the depth H is

$$H_t + Q_x = s_x, \tag{3.21}$$

where $s(x,t)$ is a prescribed surface flux function (being the space integral of the accumulation rate) and Q is the mass flux through a section.

If we set the bedrock profile $h = 0$ and neglect longitudinal stress effects ($\delta = 0$), then the basal stress τ_b is, from (3.1),

$$\tau_b = H[1-\mu H_x], \tag{3.22}$$

the flux Q is

$$Q = Hu_b + \frac{H^{n+2}}{n+2}[1-\mu H_x]^n, \tag{3.23}$$

and the basal sliding law requires

$$\tau_b = f(u_b), \quad \text{or} \quad u_b = F(\tau_b), \tag{3.24}$$

where F is the inverse of f. The neglect of δ (i.e. longitudinal stress terms) is valid provided $u_b \lesssim O(1)$ and Q (or H) does not change by $O(1)$ in a distance $O(\delta)$; this will depend on the magnitude of μ. We note from parametric estimates (Fowler & Larson 1978) that since $\mu = \delta/\epsilon^*$, ϵ^* = mean bedrock slope, it is reasonable to take

$$\delta \ll \mu \ll 1. \tag{3.25}$$

If u_b or H changes by $O(1)$ in a distance $O(\delta)$ then the discussion after (3.3) indicates that the shock structure is given by the solution of an elliptic problem, and is correspondingly messy. Alternatively, we may suppose that basal sliding is dominant, in that the longitudinal velocity may be written as

$$u = u_b(x,t) + O(\beta) \quad (\beta \ll 1); \tag{3.26}$$

then, to leading order in β, (3.23) is

$$Q = Hu_b, \tag{3.27}$$

$$\tau_b = f(u_b), \tag{3.28}$$

and a uniform asymptotic expression for the basal shear stress is

$$\tau_b = H[1-\mu H_x] + \alpha[Hu_b'|u_b'|^{1/n-1}]_x, \tag{3.29}$$

$$\alpha = 2\delta(2\nu)^{1/n}, \quad \nu = \delta/\beta \lesssim 1, \tag{3.30}$$

where $u_b' = \partial u_b/\partial x$. The term in α is a leading-order correction in regions where u_b changes rapidly over distances (from (3.11)) of $O(\delta/\beta^{1/(n+1)})$, and is present in case longitudinal stress variation is important in smoothing out discontinuities in seasonal shock waves. Away from such regions, we neglect this term because we may take $\alpha \ll 1$, typically $\alpha \sim \delta$. The assumption that $\nu \lesssim 1$, i.e. $\delta \lesssim \beta$, restricts our attention to glaciers that are 'malleable', i.e. in which shear stress does indeed balance gravity; this is a reasonable assumption to make. The set (3.21), (3.26), (3.28) and (3.29) provides a second self-consistent model, which we shall see is useful in studying seasonal waves. Additionally, the form of the function $f(u_b)$ in (3.28) must be specified.

4. Surface waves

Consider the first model set of equations derived in the preceding section:

$$\left.\begin{aligned} &H_t + Q_x = s_x, \\ &\tau_b = H[1-\mu H_x], \\ &Q = Hu_b + \frac{H^{n+2}}{n+2}[1-\mu H_x]^n, \\ &u_b = F(\tau_b). \end{aligned}\right\} \tag{4.1}$$

4.1. *The case $\mu = \alpha = 0$*

If we adopt the above approximations, and put $u_b = 0$, then

$$Q = H^{n+2}/(n+2), \tag{4.2}$$

$$H_t + H^{n+1}H_x = s_x. \tag{4.3}$$

This form of the equation was studied by Fowler & Larson (1980*b*). It may be written in characteristic form

$$\frac{dx}{dt} = H^{n+1}, \quad \frac{dH}{dt} = s_x(x,t), \tag{4.4}$$

whence the solution may be written down in terms of a characteristic parameter (see Fowler & Larson 1980*b*). We note from (4.4) that the wave speed is H^{n+1}; from (3.5) the surface speed is

$$u_s = u_b + \frac{[1-\mu H_x]^n}{n+1} H^{n+1}; \tag{4.5}$$

with $u_b = \mu = 0$, this implies that the wave speed is $n+1 \simeq 4$ times the surface speed. The characteristic solution can be used to study small disturbances, seasonal variations, and the formation and propagation of shocks. When $s \equiv s(x)$, perturbations of the equilibrium profile $H^{n+2}/(n+2) = s(x)$ will generally lead to shock formation in which the depth is discontinuous across a moving shock front $x_d(t)$, where

$$\frac{dx_d}{dt} = \frac{[Q]}{[H]} = \frac{H_1^{n+2} - H_2^{n+2}}{(n+2)(H_1 - H_2)}, \tag{4.6}$$

and H_1 and H_2 are the values of H on either side of the shock. Stable shocks are those where H decreases across a shock as x increases (Fowler & Larson 1980*b*, Whitham 1974).

The above discussion remains valid if $u_b \neq 0$, provided the sliding velocity is a well-behaved function of the basal stress. With $\mu = 0$, we have

$$\tau_b = H, \tag{4.7}$$

whence

$$u_b = F(H), \tag{4.8}$$

$$Q = Q(H) = HF(H) + H^{n+2}/(n+2). \tag{4.9}$$

The wave speed is

$$Q'(H) = (HF)' + H^{n+1}, \tag{4.10}$$

whereas the surface velocity is now

$$u_s = F(H) + \frac{H^{n+1}}{n+1}. \tag{4.11}$$

If we suppose

$$F(H) = CH^m \quad (m \leqslant n), \tag{4.12}$$

then

$$\left.\begin{aligned} Q'(H) &= (m+1)CH^m + H^{n+1}, \\ u_s &= CH^m + \frac{H^{n+1}}{n+1}, \end{aligned}\right\} \tag{4.13}$$

whence

$$(m+1)u_s \leqslant Q'(H) \leqslant (n+1)u_s. \tag{4.14}$$

If ice were Newtonian, then $m = 1$ (Nye 1969); if regelation and deformation are equally important, then an empirical estimate is $m = \frac{1}{2}(n+1)$ (Weertman 1957). Thus the wave speed is still likely to be about three to four times the surface speed, provided these values of u_b are realistic; this is the case as long as the basal ice maintains contact with the bedrock.

In essence, the results above have been derived previously (Nye 1960); they are restated here to emphasize that well-behaved functions $F(\tau_b)$ ($F \sim 1$, $F' \sim 1$) will contribute to surface waves, but not to seasonal waves.

4.2. *The case* $\mu \neq 0$, $\alpha = 0$: *shock structure*

The neglect of μ in (4.1) will remain valid as long as $H_x = O(1)$ but, when $|H_x| \sim 1/\mu \gg 1$, then this is no longer true. $|H_x|$ is large precisely where the equations with $\mu = 0$ develop shocks, and thus we hope that μ will play the role of a diffusion-like parameter for the nonlinear hyperbolic equation for H.

We first put $u_b = 0$; then (4.1) gives

$$H_t + \left\{\frac{H^{n+2}}{n+2}(1-\mu H_x)^n\right\}_x = s_x, \tag{4.15a}$$

or

$$H_t + (1-\mu H_x)^n H^{n+1} H_x = s_x + \frac{\mu n}{n+2} H^{n+2}(1-\mu H_x)^{n-1} H_{xx}. \tag{4.15b}$$

This is in the form of a convective diffusion equation, and μ does indeed play the role of a diffusion coefficient. We remark that the diffusion is *degenerate* in that the coefficient of H_{xx} goes to zero with H, and thus we expect some subtlety in prescribing boundary conditions where $H = 0$, particularly at the snout (Fowler & Larson 1980*b*, Smirnova 1963, Nye 1963).

We wish to use (4.15) to examine the structure of diffusionless shocks in the same manner that Burgers' equation (Whitham 1974) is used to study the shock structure of weak shocks in gas dynamics (Cole 1968). We note that, in a shock, a relevant length scale is $x \sim O(\mu)$, and the corresponding time scale is also $O(\mu)$. We therefore define

$$x = \mu X, \quad t = \mu\tau; \tag{4.16}$$

then, to leading order in μ, (4.15*a*) is

$$H_\tau + \left[(1-H_X)^n \frac{H^{n+2}}{n+2}\right]_X = 0. \tag{4.17}$$

Because the shock structure has length scale $x \sim \mu \ll 1$, the relevant boundary conditions to apply are that

$$H \to H_\pm, \quad X \to \pm\infty, \quad H_- > H_+, \tag{4.18}$$

where H_+ and H_- are values of H on either side of a shock when $\mu = 0$. We require also an initial condition. However, since the time scale for (4.17) is $t \sim \mu \ll 1$, the shock structure reacts very rapidly compared with the rest of the surface. We may thus suppose that H_+ and H_- are constant (vary with t and not τ), and that the solution of (4.17) rapidly evolves into a travelling-wave solution with constant profile. Note that we expect $H_X < 0$, so that $1 - H_X > 0$, as required by the original model (Fowler & Larson 1978).

We now seek a solution

$$H = \phi(z), \quad z = X - c\tau, \tag{4.19}$$

where the shock speed c is to be determined, and $\phi \to H_\pm$ as $z \to \pm\infty$. We quickly find that ϕ is given implicitly by

$$z = \int_\phi^{\phi_0} \frac{d\phi}{A(\phi)}, \quad A(\phi) = \left[\frac{(n+2)(K+c\phi)}{\phi^{n+2}}\right]^{1/n} - 1, \tag{4.20}$$

and K and c are given by

$$K = \frac{H_-^{n+2}}{n+2} - cH_- = \frac{H_+^{n+2}}{n+2} - cH_+. \tag{4.21}$$

The upper limit $\phi_0 \in (H_+, H_-)$ is arbitrary (it merely prescribes the origin of z). Consideration of $A(\phi)$ shows that (4.20) gives a monotonic trajectory between H_- and H_+, so the existence of the postulated shock structure is assured. Furthermore, this analysis carries over directly to the case $u_b \neq 0$, provided we suppose $\phi F(\phi)$ is convex, i.e. $(\phi F(\phi))'' > 0$ (which is reasonable).

The neglect of terms in δ in the above analysis is valid provided $\mu \gg \delta$, as in (3.25), that is if the mean bedrock slope $\epsilon^* \ll 1$. This is certainly a sensible estimate: one might typically have $\epsilon^* \sim 10^{-1}$. If $\beta \ll 1$ in (3.26), so that the flow is predominantly by basal slip, then the same analysis of the corresponding equations goes through: from (3.29) and (3.30), the neglect of δ is valid if

$$\alpha = 2\delta(2\nu)^{1/n} \ll \mu^{1+1/n}, \quad \nu = \delta/\beta,$$

i.e. if

$$\epsilon^* \ll \tfrac{1}{2}\beta^{1/(n+1)}. \tag{4.22}$$

This also is not unrealistic, in view of the high value of n. Since the presence of μ is sufficient to smooth out shocks in H, we might suspect that small values of α (or δ) have a regular perturbative effect on these shock structures, since they are not necessary for their existence. This conjecture is borne out in the study of seasonal waves, and shows that, from a dynamic point of view, it is surface slope and not longitudinal stress which has the principal controlling influence on surface wave motions.

The nonlinear diffusive analysis above should be put in its proper context. Other authors (Hutter 1980, Lick 1970) have sought higher-order nonlinear equations with which to study nonlinear wave motions, by analogy with the derivation of the Korteweg–de Vries equation (Cole 1968) in shallow-water theory with small disturbances. This equation governs the long-time weakly nonlinear behaviour of small-amplitude waves in shallow water. Ordinary shallow-water theory is not limited to small amplitudes, but develops shocks because of the nonlinearity. The papers of Hutter and Lick are thus both concerned with surface waves of (dimensionless) amplitude a and wavelength λ, and the implicit assumption is that $a \ll 1$, $\delta \ll \lambda \ll 1$ (so that the ice mass is virtually a slab). The analysis then purports to show that variants of Burgers' or the Korteweg–de Vries equation arise in the same way; however, complications occur because of the singularity of Glen's flow law at zero stress, and the extra parameter δ (and μ). The present theory avoids this difficulty by not being restricted to small amplitudes, and hence not encountering increasing singularities at higher order in δ. Although we imagine that such singularities would properly occur in succeeding terms of an expansion in δ, we surmise that these could be dealt with by an appropriate strained co-ordinate (Van Dyke 1975), and are thus unimportant at leading order in the present approximation. This is not necessarily true for 'weakly nonlinear' analyses, and it is therefore worth asking, in the glaciological context at least, if such analyses are relevant in view of the dominant diffusive influence of μ.

The present model describes surface perturbations of wavelength λ provided only $\lambda \gg \delta$, and the amplitude is unlimited. This suggests that the shock structure above is sufficient to cope with the evolution of any initial disturbance whatsoever. Furthermore, it is reasonable to suppose that even a very short wavelength disturbance would spread out until the shock structure previously derived was attained since the longitudinal stress also has a diffusive effect. In that case, the analysis here gives an effectively complete description of surface wave motions. The shock thickness is $O(\mu)$, or dimensionally $O(\mu l) = O(d/\epsilon^*)$, where ϵ^* is the mean bedrock slope, and d is the depth. Thus the shock thickness may range from 200 m for thin steep glaciers to 10 km for thick, gently sloping ones. Thus, even though $\mu \ll 1$, it is unlikely that severe surface slopes will actually be observed in practice.

5. The basal sliding law

We will now suppose that the flow is predominantly by basal sliding, so that $\beta \ll 1$ in (3.4). We have already noted in § 4 that, if the basal stress $f(u_b)$ is well-behaved, then we can expect kinematic surface waves to propagate downstream at low speeds. Seasonal waves, on the other hand, propagate at high speeds, but without substantial changes in depth; it therefore seems that such behaviour may be due to the sliding law *not* being well-behaved. In order to consider this further, it is necessary to digress to a discussion of sliding theory.

As previously discussed, we can expect the sliding law $\tau_b = f(u_b)$ to be a well-behaved, increasing function of u_b so long as the ice maintains effective contact (separated only by a thin water film) with the bedrock; this concurs with the intuitive notion that the frictional resistance increases as the velocity increases. There are two things that can go wrong. In the formulation of a complete sliding theory (Fowler 1981), the thickness of the regelation water film is unknown, and is to be determined as part of the problem. However, a typical estimate for this thickness is $\sim 10^{-6}$ m, and thus the ice flow is unaware of the regelation film, at least as far as a disturbance of the bedrock boundary is concerned; in other words, the ice-flow/bedrock temperature sliding problem (as formulated e.g. by Nye 1969) uncouples from the determination of the regelation film thickness. However, the dimensionless $O(1)$ film thickness $\Sigma(x)$ is then determined from the ice-flow/bedrock temperature problem, and one finds that there is no obvious reason why Σ should remain positive and finite. If either $\Sigma \to \infty$ or $\Sigma \to 0$, then we deduce that the dimensionless formulation is inconsistent, and the problem must be recast. This inconsistency has been noted by Nye (1973) (see also discussion by Fowler 1981; Morris 1976, 1979). As far as I am aware, this inconsistency has not been resolved. In the regelation context, it has been suggested (Nye 1973) that other physical features may be relevant; on the other hand, the mathematical inconsistency in sliding theory needs to be removed before introducing additional physics, so that the stated problem at least has a mathematical solution. In principle, it is fairly clear how to do this. We remove the restriction that Σ should be $O(1)$ by considering three differing boundary types: attached, lubricated and separated. While $\Sigma \sim O(1)$, the bed is lubricated, and the ice/rock problem uncouples as before; if then $\Sigma \to 0$, we expect an 'attached' region, in which there is no regelation film, and we require (for example) the shear stress at the interface to be non-zero. (Such discontinuity in the boundary conditions for a viscous fluid will lead, on a short time scale, to viscoelastic behaviour of the medium, i.e. stick–slip motion.) If, on the other hand, $\Sigma \to \infty$ (as occurs in Nye's (1973) regelation past a wire with concave portions of the boundary) then we must have a 'separated' boundary, in which the ice–water interface leaves the neighbourhood of the bedrock. This constitutes a cavity, whose boundary is unknown. From the point of view of the flow of ice, we require an extra boundary condition in this cavity; the most likely candidate is that the pressure is constant, and this additional condition should in principle determine the free boundary. Solution of this problem has been initiated by Fowler (1979*b*), and further work is currently in progress.

Alternatively, if the bedrock is such that Σ is finite, then cavities may also occur by virtue of the pressure in the water film attaining the triple-point pressure at which ice, water and water vapour can co-exist (Morland 1976). This is about 0·006 bar (Dorsey 1940); cavitation of this type is directly comparable to ordinary fluid cavitation (Batchelor 1967), and will occur if the pressure variation due to flow over the bedrock reaches the order of magnitude of the overburden pressure $\sim \rho g'd$. With a basal stress $[\tau]_0$ and a mean bedrock roughness slope ν^*, the pressure variation required to balance the drag is $\sim [\tau]_0/\nu^* = \rho g'd\epsilon^*/\nu^*$ by definition of $[\tau]_0$ (A 18). Thus cavitation will occur if $\rho g'd\epsilon^*/\nu^* \gtrsim \rho g'd$, i.e. if the mean bedrock slope $\epsilon^* >$ the mean bedrock roughness slope ν^* (Morland 1976); thus cavitation by this means is more likely to occur in steep glaciers with smooth bedrocks.

Sliding with cavitation

There are at least two possible mechanisms, described above, that will predict subglacial cavities; and such cavities are indeed observed (e.g. Theakstone 1979; Vivian & Bocquet 1973). It is thus pertinent to enquire what the effect of such cavities will be on the drag. The unsatisfactory state of the regelation theory is probably partly to blame for the fact that few coherent examinations of mathematical formulations of this problem have been attempted. The main exponent of sliding with cavitation has been Lliboutry, who has, in a series of papers (1968, 1976, 1978, 1979) attempted to devise tractable forms for the sliding law when cavities are present. His papers do not solve mathematical problems, but they possess great physical insight into the process of cavitation, and give important clues as to the possible behaviour. Lliboutry (1968) and Weertman (1964) detected the possibility of a *multiple*-valued sliding law $F(\tau_b)$; with $\tau_b = f(u_b)$, this would mean f would decrease with increasing u_b over a range of velocities. It is obvious that, at a given velocity, the drag will be less when cavitation occurs than when it does not; this is because (at least for cavities in which the triple point pressure is reached) the cavity pressure is constant *downstream* of obstacles, and thus the integral of the pressure over the ice interface (which is the drag) only has an effect due to the pressure on the upstream face. Thus the effect of cavitation must be to decrease $f'(u_b)$; the extent of this decrease is uncertain.

Fowler (1979*b*) gives the result of a computation which shows that, for the particular case of a Newtonian flow over a sinusoidal bedrock with $\nu^* \to 0$ (vanishing bedrock roughness slope), the formation of a cavity at a critical stress decreases f' to zero; thus the prospect of f' becoming very small or even negative seems plausible.

The experimental results of Drake & Shreve (1973) are of relevance here. They found that regelation velocities of wires through ice blocks increased dramatically (by factors of 10^3) when the drag on the wire reached approximately 1 bar (i.e. the ambient hydrostatic pressure). At this level of the drag one supposes that a cavity is able to

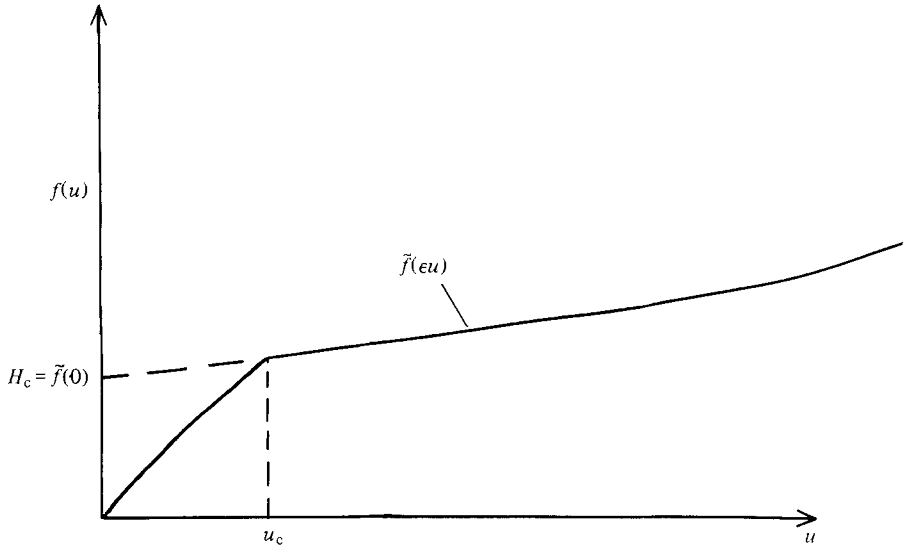

FIGURE 2. Assumed form of sliding law in this paper; for $u \geqslant u_c, f(u) = \tilde{f}(\epsilon u)$, where $\tilde{f}(\phi)$ is regular at $\phi = 0$.

form, thus enabling the velocity to increase dramatically for small increases of the driving load. Their experimental results are very similar to figure 2 above.

Lliboutry, in his latest paper on this subject (1979), gives a good review of various relevant physical mechanisms, together with an approximate derivation of an appropriate sliding law. He distinguishes two modes of cavitation, autonomous and interconnected. Autonomous cavitation occurs when each cavity is isolated, and particularly when it does not connect up to the subglacial hydraulic system (which drains off melt water). In this case the pressure in a cavity is determined by the requisite mathematical description; for cavities at triple point pressure, this pressure is essentially zero. In the *interconnected* regime, the cavities are connected via striae, joints or other channels in the bedrock, and the pressure in the cavities is consequently essentially that of the local subglacial drainage system, p_w (with insignificant head loss between cavities and the main channel system). This concept has a slight subtlety built into it: if the entire bedrock is so scratched that the water pressure everywhere is p_w then presumably not only cavities but also the water film must be at pressure p_w. This appears to contradict the whole notion of a water film, which requires a pressure gradient to drive the regelation flow. A reformulation of the problem would be to consider an ice boundary which is alternatively attached (no water film) or separated, in which case we prescribe no shear stress *and* $p = p_w$ on the separated boundary. Such a formulation is again fraught with difficulty.

We have discussed the physics of sliding in some detail, in order to explain how cavitation may lead to a sliding stress $f(u)$ which is radically different to that appropriate when no ice–bedrock separation occurs. Clearly, this is a subject of continuing research, and it is not our intention to try and rigorously derive an appropriate law here. Rather, we seek to study the dynamic effect of choosing an $f(u)$ that reflects the behaviour discussed above. In the analysis to follow, we shall therefore specifically assume that the sliding law $f(u)$ has the form shown in figure 2. That is, we shall suppose that, for u greater than some critical velocity u_c at which cavitation sets in, f' is small and f increases slowly as shown. For $u < u_c, f' \sim O(1)$; specifically we suppose $f'(u_c-) \simeq m$ (to $O(\epsilon)$) so that the sliding law valid for both $u > u_c$ and $u < u_c$, $|u-u_c| \ll 1$, may be written in the form

$$f(u) = \tilde{f}(\epsilon u) + m(u-u_c)\mathscr{H}(u_c - u), \tag{5.1}$$

where $\mathscr{H}(\phi) = 1$ if $\phi > 0$, $\mathscr{H}(\phi) = 0$ if $\phi < 0$, denotes the Heaviside step function, and $\tilde{f}$ is an $O(1)$ function of its $O(1)$ argument. The assumption that f' is discontinuous at $u = u_c$ is not unrealistic and, in any case, is not expected to have a major effect on the solutions. We suppose that $\tilde{f}$ can be continued back to $u = 0$ as shown in the dotted line in figure 2, so that we may take Taylor expansions about $u = 0$; we suppose $\tilde{f}, \tilde{f}'$, etc. are $O(1)$. The parameter ϵ is chosen $\ll 1$ to reflect the possibility that cavitation has this drastic effect on the sliding law. As already stated, results of Drake & Shreve (1973) and Fowler (1979*b*) support this conjecture. We do not propose (5.1) or figure 2 as a universal form of sliding law; we simply observe that it is plausible, and examine the dynamic consequences. Before proceeding, we note that a sliding law of this form is probably reasonable if we can neglect the effect of p_w. This, however, is an unrealistic assumption; in particular, it is known that seasonal fluctuations (due to precipitation) can affect the sliding velocity via the variation in p_w. In a more realistic analysis we should take account of this. Nevertheless, the present work offers a first consistent analysis of the dynamic effects of a cavitational sliding law, and as such we wish to lay our emphasis more on the dynamics, rather than become enmeshed in the complexities of the particular sliding law.

6. Seasonal waves

In this section, we follow the analogous procedure of § 4 in studying seasonal waves. Thus, first we identify the basic diffusionless wave-propagation characteristics, and then we seek to understand how discontinuities in the predicted flow may be smoothed by appropriate diffusion parameters. The equations (3.21), (3.27)–(3.29) are

$$H_t + (Hu)_x = s_x, \tag{6.1a}$$

$$H[1-\mu H_x] + \alpha[Hu_x|u_x|^{1/n-1}]_x = f(u). \tag{6.1b}$$

For convenience, we take $n = 1$ in (6.1*b*); this simplifies the analysis, without (we hope) altering the qualitative conclusions. We make some comments on length scales as we proceed. With $n = 1$, $\alpha = 4\delta^2/\beta$ is not very much different from its value when $n = 3$, say. Then (6.1*b*) is

$$H[1-\mu H_x] + \alpha(Hu_x)_x = f(u). \tag{6.2}$$

The parameters μ and α are included in (6.2) to represent the effects of diffusion, due respectively to large surface gradients and large longitudinal velocity gradients. We study first the diffusionless equations.

6.1. $\mu = \alpha = 0; \epsilon \to 0$

With $Q = Hu$, the steady-state solution of (6.1) is $Q = s(x)$, where the flux function $s(x)$ is positive and typically concave downwards between x_0 (the head) and x_S (the snout) respectively. If we put $\mu = \alpha = 0$, then this implies that

$$H = f(u), \quad uf(u) = s(x). \tag{6.3}$$

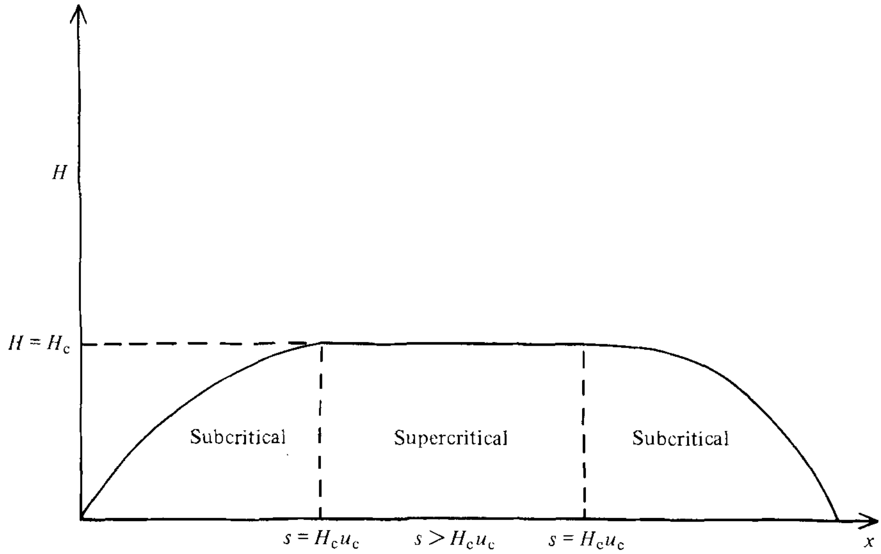

FIGURE 3. Schematic profile of steady-state glacier surface showing subcritical ($u < u_c$) and supercritical ($u > u_c$) zones.

For $u < u_c$, f is a well-behaved function of u, and so u and hence H in these regions both increase in a smooth way with s; in these regions, $s \lesssim u_c f(u_c)$. If (as shown in figure 2) we define

$$H_c = \tilde{f}(0), \tag{6.4}$$

then $H \lesssim H_c$ when $u \lesssim u_c$ in these regions. However, if s increases beyond $H_c u_c$ (the critical flux), then cavitation is initiated and, as s increases, figure 2 shows that u increases without an appreciable change of H; thus, while $s \sim O(1)$, $s \gtrsim H_c u_c$, we have $u \sim O(1)$, $H \simeq H_c$, so that the steady state is approximately

$$H \simeq H_c, \quad u \simeq s(x)/H_c. \tag{6.5}$$

A typical longitudinal profile of H is then as shown in figure 3.

In subcritical regions ($H \lesssim H_c$), kinematic waves propagate as before, but the critical zones ($H \simeq H_c$) admit the possibility of seasonal waves. To examine these, let $x = 0$ at the upstream end of a critical zone, i.e. define x so that $s(0) = H_c u_c$. For $x > 0$, $u > u_c$, and $f(u) = \tilde{f}(\epsilon u)$; note that $H_c \neq f(u_c)$, as seen from figure 2. As long as $u > u_c$, we suppose $\tilde{f}$ has a regular Taylor series about $u = 0$. In $x > 0$, we put

$$H = H_c + \epsilon\chi, \quad t = \epsilon\tau, \quad f(u) = \tilde{f}(\epsilon u); \tag{6.6}$$

then (6.1) is, with $\mu = \alpha = 0$,

$$\chi_\tau + [(H_c + \epsilon\chi)u]_x = s_x, \tag{6.7a}$$

$$\chi = \frac{1}{\epsilon}[\tilde{f}(\epsilon u) - H_c]. \tag{6.7b}$$

If we consider variations of u in which $u \sim 1$, then, since $H_c = \tilde{f}(0)$, we expand (6.7b) to $O(\epsilon)$ as

$$\chi = u\tilde{f}'(0) + O(\epsilon), \tag{6.8}$$

so that (6.7) gives the wave equation for u, to leading order:

$$\tilde{f}'(0)\, u_\tau + \tilde{f}(0)\, u_x = s_x. \tag{6.9}$$

This is a linear wave equation for u which we can solve for arbitrary initial conditions on u. At higher order in ϵ, the full equation (from (6.7)) is weakly nonlinear; owing to the finiteness of x, however, this nonlinearity is irrelevant in considering the evolution of smooth profiles. For example, the solution of the initial-value problem when the climatic input $s(x)$ is steady, is

$$u = \frac{s(x)}{\tilde{f}(0)} + \phi\left[x - \frac{\tilde{f}(0)\,\tau}{\tilde{f}'(0)}\right], \tag{6.10}$$

which represents a rightward-travelling wave with dimensionless speed

$$dx/dt = \epsilon^{-1}\, dx/d\tau = \tilde{f}(0)/\epsilon \tilde{f}'(0).$$

Note that we are here assuming $\tilde{f}'(0) > 0$ (figure 2). If the sliding law was multivalued, we should have $\tilde{f}'(0) < 0$, and thus *backward*-travelling waves; this is exactly what must happen in surges (see Meier & Post 1969, Meier 1979). With $\tilde{f}'(0) > 0$, ϕ in (6.10) is the perturbative term. If this arises from an initial fluctuation in H of $O(\epsilon)$, then outside the critical zone, conditions are essentially constant in time; it follows that we require the perturbation to become small at critical boundaries, where $u = u_c$. For a forward-travelling wave as in (6.10), this is done by prescribing $\phi = 0$ at $x = 0$: in fact $\phi(\xi) = 0$ for $\xi < 0$. Any $O(1)$ disturbance to u propagates downstream, and the velocity returns to its equilibrium value. Fowler (1979*b*) derived an expression for u when periodic seasonal fluctuations in accumulation contribute to seasonal waves. The appropriate boundary condition is to have $u = u_c$ at $x = 0$. However, we also require that $u = u_c$ at the x-value downstream (x_1, say) where the steady-state flux $s(x)$ becomes critical, i.e. $s(x_1) \simeq H_c u_c$. The solution of (6.10) cannot do this, and thus a more-detailed consideration of the model is necessary to complete the analysis.

Let us examine the main characteristics of these waves. *In this theory* (i.e. with the various assumptions we have made) seasonal waves will travel at velocities $O(1/\epsilon)$ times the surface speed in critical zones, where the depth H is effectively constant. The fluctuation in velocity u will generally (even for a small climatic fluctuation) be $O(1)$, whereas, although there is a corresponding (and proportional, (6.8)) surface wave, its amplitude is much smaller. Observations indicate a range for ϵ of 0·005–0·05 (speeds 20 to 200 times the surface speed). Such a prediction is in principle verifiable by a corresponding study of cavitational sliding.

We observe that the analysis of seasonal waves by Fowler (1979*b*), while in essence giving similar results, was derived in a more heuristic manner, by assuming *a priori* that $Q'(H)$ was essentially constant ($= 1/\epsilon$ there). Here we see that

$$\frac{dQ}{dH} = \left[\frac{d}{du}(uf(u))\right] \Big/ \left[\frac{d}{du}(f(u))\right] = [u\tilde{f}(\epsilon u)]'/\epsilon \tilde{f}'(\epsilon u) \simeq \tilde{f}(0)/\epsilon \tilde{f}'(0) + O(1), \tag{6.11}$$

and thus this assumption is in fact borne out in the present case.

6.2. *Shock formation: transcritical shocks*

As remarked above, the linear wave equation (6.9) is derived as $\epsilon \to 0$ from a weakly nonlinear equation (6.7). This implies that, as long as $u > u_c$, a smooth initial profile will remain smooth on times of $O(1)$ (i.e. of interest), since the nonlinearity, which has the effect of steepening the profile, is small. Nevertheless, it is of interest to compare the relative effects of nonlinearity and diffusion (with both small) in case we wish to analyse the development of a discontinuous initial profile such as might be physically motivated by the surge of a tributary glacier. The procedure is similar to that in § 4, but we omit the details.

Consider a shock structure joining two regions of the supercritical regime, $u > u_c$. Within the shock we rescale x with a parameter σ, to be determined. We retain $\epsilon \neq 0$ since we will require nonlinearity in order to balance the diffusive terms. It turns out from (6.1) that the nonlinearity is actually $O(\epsilon^2)$, and the relative magnitudes of terms in the travelling-wave equation for u (analogous to (4.20)) are:

surface-slope diffusion, $\mu\epsilon/\sigma$;

longitudinal-stress diffusion, $\alpha/\sigma^{1+1/n}$;

nonlinearity, ϵ^2.

A study of the actual system shows that each diffusion term separately (or together) will provide a shock structure, provided u decreases with x through the shock (as suggested physically). Choice of σ is determined by balancing nonlinearity with the largest diffusive term. For example, take $n = 1, \alpha \sim 10^{-3}, \mu \sim 10^{-1}, \epsilon \sim 10^{-2}$; then we choose $\alpha/\sigma^2 = \epsilon^2$, hence $\sigma = \alpha^{\frac{1}{2}}/\epsilon$. However, $\alpha^{\frac{1}{2}}/\epsilon \sim 3 \gtrsim O(1)$ in this case, and the 'shock' length is therefore not in fact small. The conclusion is that, on length scales $\lesssim O(1)$, the nonlinearity of the (supercritical) sliding law is far too small (if $\epsilon \lesssim 10^{-2}$) to maintain a coherent shock, and a discontinuous initial profile simply begins to diffuse away.

Although supercritical shocks are of little concern, *transcritical shocks* are of much more interest. These must form when a forward-travelling seasonal wave in $u > u_c$ reaches the 'barrier' between super- and subcritical zones, at $u = u_c$. At this point, (5.6) shows that the characteristic slopes are discontinuous, and a transcritical shock must form, joining an upstream region in which $u > u_c$ to a downstream one in which $u < u_c$. We now analyse the propagation and decay of such shocks.

Let us first re-define the origin so that it is near the (steady-state) critical barrier where $u = u_c$. Then $H \simeq H_c$, and to be specific we define

$$s(0) = H_c u_c, \tag{6.12}$$

with $s(x) \gtrless 0$ in $x \lessgtr 0$. Rather than put $\alpha = \mu = 0$ immediately, we observe that a shock is essentially a singular region joining regions where μ and α have a regular (small) perturbative effect on the solutions. From the definitions of α, μ and ϵ, we take reasonable estimates of these parameters to be $\alpha \sim 10^{-3}$, $\mu \sim 10^{-1}$, $\epsilon \sim 10^{-2}$. It is thus natural to define

$$\alpha = \epsilon^{\frac{3}{2}}\tilde{\alpha}, \quad \mu = \epsilon^{\frac{1}{2}}\tilde{\mu}, \tag{6.13}$$

and to consider $\tilde{\alpha}$ and $\tilde{\mu} = O(1)$. It turns out that the definitions in (6.13) are also convenient distinguished limits in later manipulation.

In order to formulate a coherent mathematical problem, we will examine the effect of a seasonal wave arriving at the critical barrier $x \simeq 0$, and travelling into a subcritical zone in the steady state. Additionally, we take $n = 1$ in (6.1). With $\partial/\partial t = 0$ and $f(u)$ given by (5.1) for $u < u_c$, and using (6.13), the equations (6.1) imply

$$uH = s(x), \tag{6.14}$$

$$H[1-\tilde{\mu}\epsilon^{\frac{1}{2}}H_x] = H_c + m(u-u_c) + O(\epsilon), \tag{6.15}$$

where the expansion for $f(u)$ in (6.15) is valid to $O((u-u_c)^2)$ if $|u-u_c| \ll 1$, whatever the form of $f(u)$ at smaller u. The subcritical *regular* solutions of (6.14) and (6.15) are given by power series in $\epsilon^{\frac{1}{2}}$, and may be found by straight substitution. We suppose $s'(0) = -a$ $(a > 0)$; then, for small x, one finds

$$H \sim H_c - \frac{max}{H_c + mu_c} - \epsilon^{\frac{1}{2}}\frac{\tilde{\mu}maH_c^2}{(H_c+mu_c)^2} + O(\epsilon, \epsilon^{\frac{1}{2}}x, x^2), \tag{6.16}$$

$$u \sim u_c - \frac{ax}{H_c + mu_c} + \epsilon^{\frac{1}{2}}\frac{\tilde{\mu}maH_c u_c}{(H_c+mu_c)^2} + O(\epsilon, \epsilon^{\frac{1}{2}}x, x^2), \tag{6.17}$$

as may be checked *a posteriori*.

The incoming seasonal wave is essentially given by (6.8) and (6.10), even when $\mu \sim \epsilon^{\frac{1}{2}}$. Specifically, when $u > u_c$,

$$\left.\begin{aligned} H &\sim H_c + O(\epsilon), \\ u &\sim u_c + \Phi(\tau) + O(\epsilon^{\frac{1}{2}}, x), \end{aligned}\right\} \tag{6.18}$$

where in terms of (6.10) we have put $\Phi(\tau) = \phi[-\tilde{f}(0)\tau/\tilde{f}'(0)]$; $\Phi(\tau)$ represents the (positive) difference between u and u_c of the incoming seasonal wave at the critical barrier $x \simeq 0$. For an $O(1)$ disturbance to the (finite) critical zone, we will have $\Phi = 0$ for $\tau < 0$, and positive over a finite range, $(0, M)$ say, and then $\Phi = 0$ for $\tau > M$.

Now let $x_d(\tau)$ denote the position of a transcritical shock. The values of H and u on x_d+ and x_d- are given by (6.16) and (6.17), and (6.18), respectively, provided x_d is small; this will be the case, at least for small τ. The evolution of the shock with time is given by

$$\frac{dx_d}{d\tau} = \frac{\epsilon[uH]_+^-}{[H]_+^-} \tag{6.19}$$

(see e.g. Murray 1970), where the square brackets denote the jump in the enclosed quantity from x_d+ to x_d-. From (6.16)–(6.18), we therefore find that

$$\frac{dx_d}{d\tau} = \frac{\epsilon[\{H_c+O(\epsilon)\}\{u_c+\Phi+O(\epsilon^{\frac{1}{2}},x_d)\} - \{H_c u_c + O(x_d, \epsilon^{\frac{1}{2}})\}]}{\{H_c+O(\epsilon)\} - \left\{H_c - \dfrac{max_d}{H_c+mu_c} - \epsilon^{\frac{1}{2}}\dfrac{\tilde{\mu}maH_c^2}{(H_c+mu_c)^2} + O(\epsilon, \epsilon^{\frac{1}{2}}x_d, x_d^2)\right\}}. \tag{6.20}$$

For an incoming seasonal wave, a transcritical shock first forms where $u = u_c$, whence (6.17) appears to imply (but see below) that the initial condition for x_d is

$$x_d(0) = \epsilon^{\frac{1}{2}}\frac{\tilde{\mu}mH_c u_c}{H_c+mu_c}; \tag{6.21}$$

this confirms that $x_d \sim \epsilon^{\frac{1}{2}} \ll 1$, at least for sufficiently small τ. As long as $x_d \gtrsim O(\epsilon^{\frac{1}{2}})$, we put

$$x_d = \epsilon^{\frac{1}{2}}x_d^*, \quad x_d^* \sim O(1), \tag{6.22}$$

and then (6.20) is, correct to $O(\epsilon^{\frac{1}{2}})$,

$$\frac{dx_{\mathrm{d}}^*}{d\tau}=\frac{H_{\mathrm{c}}\,\Phi(\tau)}{\dfrac{ma}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\left[x_{\mathrm{d}}^*+\dfrac{\tilde{\mu}H_{\mathrm{c}}^2}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\right]},\quad x_{\mathrm{d}}^*(0)=\frac{\tilde{\mu}mH_{\mathrm{c}}u_{\mathrm{c}}}{H_{\mathrm{c}}+mu_{\mathrm{c}}};\qquad (6.23a,b)$$

it follows from (6.23) and the nature of $\Phi(\tau)$, that $x_{\mathrm{d}}^*\sim 1$ for all τ, so that (6.23) is accurate for all τ. The solution of (6.23) is

$$\left[x_{\mathrm{d}}^*+\frac{\tilde{\mu}H_{\mathrm{c}}^2}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\right]^2=(\tilde{\mu}H_{\mathrm{c}})^2+\frac{2H_{\mathrm{c}}(H_{\mathrm{c}}+mu_{\mathrm{c}})}{ma}\int_0^\tau\Phi(\tau)\,d\tau.\qquad (6.24)$$

Φ is positive and of compact support, so that (6.24) shows that the shock travels forward at a speed $dx_{\mathrm{d}}/d\tau=O(\epsilon^{\frac{1}{2}})$. As τ increases, the incoming wave amplitude $\Phi(\tau)$ increases, and the shock velocity increases proportionally, though this is offset by the shock advance into subcritical territory. For Φ of compact support, $\int_0^\tau\Phi(\tau)\,d\tau$ is bounded above, and so $dx_{\mathrm{d}}^*/d\tau\rightarrow 0$ as $\tau\rightarrow M>0$ $(\Phi(M)=0)$. We then expect that neglected terms $O(\epsilon^{\frac{1}{2}})$ in (6.20) are relevant, and thus that, for $\tau>M$, the shock continues to propagate as a subcritical kinematic wave of amplitude $O(\epsilon^{\frac{1}{2}})$. Since in fact the transition shock width (as we show below) is $O(\epsilon^{\frac{1}{2}})$ if α, μ are given by (6.13), and the shock width of a kinematic wave is also $O(\mu)\sim O(\epsilon^{\frac{1}{2}})$, we anticipate that the profile of the wave will evolve smoothly; only its speed will change quite abruptly from $dx/dt\sim O(\epsilon^{-\frac{1}{2}})$ to $dx/dt\sim O(1)$.

Although (6.23a) is a valid representation for the shock speed, the initial condition is less accurate. This is because the steady-state regular solutions do not join at $u=u_{\mathrm{c}}$; thus $u<u_{\mathrm{c}}$ when $x>\epsilon^{\frac{1}{2}}\tilde{\mu}mH_{\mathrm{c}}u_{\mathrm{c}}/(H_{\mathrm{c}}+mu_{\mathrm{c}})$ (from (6.17)) but, when $u>u_{\mathrm{c}}$, we have in the steady state $H\sim H_{\mathrm{c}}+O(\epsilon)$, and thus $u\sim s(x)/H\sim u_{\mathrm{c}}-ax/H_{\mathrm{c}}$ whence $u>u_{\mathrm{c}}$ for $x<0$. Thus even the steady state is singular at $O(\epsilon^{\frac{1}{2}})$ and we require diffusive terms to cross the barrier between $x=0$ and $x=\epsilon^{\frac{1}{2}}\mu mH_{\mathrm{c}}u_{\mathrm{c}}/(H_{\mathrm{c}}+mu_{\mathrm{c}})$. The present problem thus differs from other shock-propagation problems in that the basic steady-state has a singular region incorporated in it; this is the 'barrier' between super- and subcritical zones. In view of these comments, it is only consistent to append, as initial condition to (6.23a), the requirement

$$x_{\mathrm{d}}^*(0)\leqslant\frac{\tilde{\mu}mH_{\mathrm{c}}u_{\mathrm{c}}}{H_{\mathrm{c}}+mu_{\mathrm{c}}},\quad\text{or}\quad x_{\mathrm{d}}^*(0)\sim O(1).\qquad (6.25)$$

6.3. *Transcritical diffusion*

Having shown how a shock can transmit the seasonal wave through the critical barrier, we wish to examine how diffusion will act to smooth out such a profile in reality. Let us first recall the equations to be solved. These are

$$H_t+(uH)_x=s'(x),\qquad (6.26)$$

$$H[1-\tilde{\mu}\epsilon^{\frac{1}{2}}H_x]+\tilde{\alpha}\epsilon^{\frac{3}{2}}(Hu_x)_x=\tilde{f}(\epsilon u)+m(u-u_{\mathrm{c}})\mathscr{H}(u_{\mathrm{c}}-u).\qquad (6.27)$$

The form of (6.27) shows that we can expect to have H_x and u_{xx} continuous at $u=u_{\mathrm{c}}$, but that higher derivatives will be discontinuous. To focus our attention on the transition region, we must rescale the variables. The nature of the shock solution immediately suggests that we put

$$x=\epsilon^{\frac{1}{2}}X,\quad t=\epsilon\tau;\qquad (6.28)$$

additionally we will define the critical point $X_{\mathrm{d}}(\tau)$ via

$$u = u_{\mathrm{c}} \quad \text{on} \quad X = X_{\mathrm{d}}(\tau). \tag{6.29}$$

We expect, but do not assume, X_{d} to be similar to x_{d}^{*} of (ii). The scales for u and H are a little complex, differing in $X > X_{\mathrm{d}}$ and $X < X_{\mathrm{d}}$, and also depending on whether we consider the steady or time-dependent state. With this warning, we shall try to indicate the procedure in a logical fashion. We assume that $u \gtrless u_{\mathrm{c}}$ for $X \lessgtr X_{\mathrm{d}}$.

6.3.1. *Matching*

Matching principles (Cole 1968) imply that the transition solutions should satisfy, from (6.16) and (6.17),

$$H \sim H_{\mathrm{c}} - \epsilon^{\frac{1}{2}}\left[\frac{maX}{H_{\mathrm{c}}+mu_{\mathrm{c}}} + \frac{\tilde{\mu}maH_{\mathrm{c}}^{2}}{(H_{\mathrm{c}}+mu_{\mathrm{c}})^{2}}\right] + O(\epsilon), \tag{6.30a}$$

$$u \sim u_{\mathrm{c}} + \epsilon^{\frac{1}{2}}\left[\frac{-aX}{H_{\mathrm{c}}+mu_{\mathrm{c}}} + \frac{\tilde{\mu}maH_{\mathrm{c}}u_{\mathrm{c}}}{(H_{\mathrm{c}}+mu_{\mathrm{c}})^{2}}\right] + O(\epsilon), \tag{6.30b}$$

as $X \to +\infty$, since the shock travels into a subcritical steady-state zone. To match the supercritical zone, we note from (6.27) that $H \sim H_{\mathrm{c}} + \epsilon \bar{f}'(0)\,u + O(\epsilon^{\frac{3}{2}})$ is the regular solution for it when $u > u_{\mathrm{c}}$, and thus (6.10) is still accurate when $\mu \sim \epsilon^{\frac{1}{2}}$, to $O(\epsilon^{\frac{1}{2}})$ (actually, to $O(\epsilon^{\frac{1}{2}}x)$); expanding for small x, it easily follows that we require the matching conditions

$$u \sim u_{\mathrm{c}} + \Phi(\tau + O(\epsilon^{\frac{1}{2}})) - \frac{\epsilon^{\frac{1}{2}}aX}{H_{\mathrm{c}}} + O(\epsilon), \tag{6.31a}$$

$$H \sim H_{\mathrm{c}} + O(\epsilon), \tag{6.31b}$$

as $X \to -\infty$; this includes both steady-state ($u \sim u_{\mathrm{c}} - \epsilon^{\frac{1}{2}}aX/H_{\mathrm{c}} + O(\epsilon)$) and time-dependent ($u \sim u_{\mathrm{c}} + \Phi(\tau) + O(\epsilon^{\frac{1}{2}})$) cases.

6.3.2. *Scaling*

(i) $X > X_{\mathrm{d}}$. We define, motivated by (6.30) (and foresight),

$$u = u_{\mathrm{c}} + \epsilon^{\frac{1}{2}}\left[\frac{-aX}{H_{\mathrm{c}}+mu_{\mathrm{c}}} + \frac{\tilde{\mu}maH_{\mathrm{c}}u_{\mathrm{c}}}{(H_{\mathrm{c}}+mu_{\mathrm{c}})^{2}} + \phi^{+}\right], \tag{6.32a}$$

$$H = H_{\mathrm{c}} + \epsilon^{\frac{1}{2}}\left[\frac{-maX}{H_{\mathrm{c}}+mu_{\mathrm{c}}} - \frac{\tilde{\mu}maH_{\mathrm{c}}^{2}}{(H_{\mathrm{c}}+mu_{\mathrm{c}})^{2}} + \chi^{+}\right], \tag{6.32b}$$

and we require

$$\phi^{+}, \chi^{+} \sim O(\epsilon^{\frac{1}{2}}) \quad \text{as} \quad X \to \infty, \tag{6.33a}$$

$$\phi^{+} = \frac{aX_{\mathrm{d}}}{H_{\mathrm{c}}+mu_{\mathrm{c}}} - \frac{\tilde{\mu}maH_{\mathrm{c}}u_{\mathrm{c}}}{(H_{\mathrm{c}}+mu_{\mathrm{c}})^{2}} \quad \text{on} \quad X = X_{\mathrm{d}}. \tag{6.33b}$$

Substituting (6.32) into (6.26) and (6.27), we obtain, after some arithmetic,

$$\epsilon^{-\frac{1}{2}}\chi_{\tau}^{+} + H_{\mathrm{c}}\phi_{X}^{+} + u_{\mathrm{c}}\chi_{X}^{+} = O(\epsilon^{\frac{1}{2}}), \tag{6.34a}$$

$$\chi^{+} - \tilde{\mu}H_{\mathrm{c}}\chi_{X}^{+} + \tilde{\alpha}\epsilon^{\frac{1}{2}}H_{\mathrm{c}}\phi_{XX}^{+} = m\phi^{+} + O(\epsilon^{\frac{1}{2}}). \tag{6.34b}$$

Equation (6.34) apparently implies $\chi^{+} = O(\epsilon^{\frac{1}{2}})$, whence also $\phi^{+} = O(\epsilon^{\frac{1}{2}})$, as is required at $X \to +\infty$. However, we require (at least in the unsteady case) $\phi^{+} \sim O(1)$ on $X = X_{\mathrm{d}}$; there is therefore a singular region near X_{d}, which we analyse by putting

$$X = X_{\mathrm{d}}(\tau) + \epsilon^{\frac{1}{2}}\tilde{X}. \tag{6.35}$$

This recovers the singular ϕ^+_{XX} term in (6.34b); there is no singular behaviour for χ^+, however, which is indeed $O(\epsilon^{\frac{1}{2}})$, as predicted. With (6.35), (6.34a) gives an equation for $\chi^+/\epsilon^{\frac{1}{2}}\,(\sim O(1))$, with a bounded solution provided $X_{\mathrm{d}}'(\tau) > 0$. With $\chi^+ \sim O(\epsilon^{\frac{1}{2}})$, (6.34$b$) then gives for ϕ^+, correct to $O(\epsilon^{\frac{1}{2}})$,

$$\tilde{\alpha} H_{\mathrm{c}} \phi^+_{\tilde{X}\tilde{X}} = m\phi^+. \tag{6.36}$$

Solving this, we finally have the (time-dependent) solution in $X > X_{\mathrm{d}}$, to satisfy (6.36) and (6.33),

$$\phi^+ = \frac{a}{H_{\mathrm{c}} + mu_{\mathrm{c}}}\left[X_{\mathrm{d}} - \frac{\tilde{\mu} m H_{\mathrm{c}} u_{\mathrm{c}}}{H_{\mathrm{c}} + mu_{\mathrm{c}}}\right] \exp\left[-\left(\frac{m}{\tilde{\alpha} H_{\mathrm{c}}}\right)^{\frac{1}{2}} \left(\frac{X - X_{\mathrm{d}}(\tau)}{\epsilon^{\frac{1}{4}}}\right)\right], \tag{6.37a}$$

$$\chi^+ = O(\epsilon^{\frac{1}{2}}). \tag{6.37b}$$

In the steady state, with $\partial/\partial\tau = 0$ in (6.34), $\chi^+ = -H_{\mathrm{c}}\phi^+/u_{\mathrm{c}}$, and the resulting linear equation for ϕ^+ has zero as the only bounded solution; hence the steady-state solutions are

$$\phi^+ \sim \chi^+ \sim O(\epsilon^{\frac{1}{2}}), \quad X > X_{\mathrm{d}}, \tag{6.38}$$

and, as a result, the steady-state value of X_{d} must be

$$X_{\mathrm{d}}(0) = \frac{\tilde{\mu} m H_{\mathrm{c}} u_{\mathrm{c}}}{H_{\mathrm{c}} + mu_{\mathrm{c}}}; \tag{6.39}$$

note that this concurs with the value of $x_{\mathrm{d}}^*(0)$ in (6.23b).

(ii) $X < X_{\mathrm{d}}$. Now consider the supercritical region, in which $u > u_{\mathrm{c}}$; we write

$$u = u_{\mathrm{c}} + \phi^-, \quad H = H_{\mathrm{c}} + \epsilon^{\frac{1}{2}}\chi^-; \tag{6.40}$$

together with (6.28), (6.26) and (6.27) become, with $f(u) = \tilde{f}(\epsilon u) = H_{\mathrm{c}} + \epsilon \tilde{f}'(0)\,u + O(\epsilon^2)$,

$$\chi^-_\tau + H_{\mathrm{c}}\phi^-_X = -\epsilon^{\frac{1}{2}}[a + \{\chi^-(u_{\mathrm{c}} + \phi^-)\}_X] + O(\epsilon), \tag{6.41}$$

$$\chi^- - \tilde{\mu} H_{\mathrm{c}} \chi^-_X + \tilde{\alpha} H_{\mathrm{c}} \phi^-_{XX} = O(\epsilon^{\frac{1}{2}}). \tag{6.42}$$

The time-dependent problem requires, from (6.31),

$$\phi^- \sim \Phi(\tau), \quad \chi^- \sim 0 \quad \text{to} \quad O(\epsilon^{\frac{1}{2}}) \quad \text{as} \quad X \to -\infty. \tag{6.43}$$

At leading order, ϕ^- and χ^- satisfy

$$\chi^-_\tau + H_{\mathrm{c}}\phi^-_X = 0, \tag{6.44a}$$

$$\chi^- - \tilde{\mu} H_{\mathrm{c}} \chi^-_X + \tilde{\alpha} H_{\mathrm{c}} \phi^-_{XX} = 0; \tag{6.44b}$$

we return to this linear problem in due course.

The steady-state problem requires, from (6.31),

$$\phi^- \sim -\frac{\epsilon^{\frac{1}{2}} aX}{H_{\mathrm{c}}} + O(\epsilon), \quad \chi^- \sim O(\epsilon^{\frac{1}{2}}) \quad \text{as} \quad X \to -\infty; \tag{6.45}$$

with $\chi^-_\tau = 0$ in (6.41), $\phi^-_X \sim O(\epsilon^{\frac{1}{2}})$, the solution for ϕ^- is then

$$\phi^- \sim -\frac{\epsilon^{\frac{1}{2}}}{H_{\mathrm{c}}}[aX + u_{\mathrm{c}}\chi^-] + O(\epsilon). \tag{6.46}$$

It follows from (6.42) and (6.43) that χ^- is given by

$$\chi^- = -A \exp[X/\tilde{\mu} H_{\mathrm{c}}], \tag{6.47}$$

where the constant A is to be determined by continuity requirements at X_{d}.

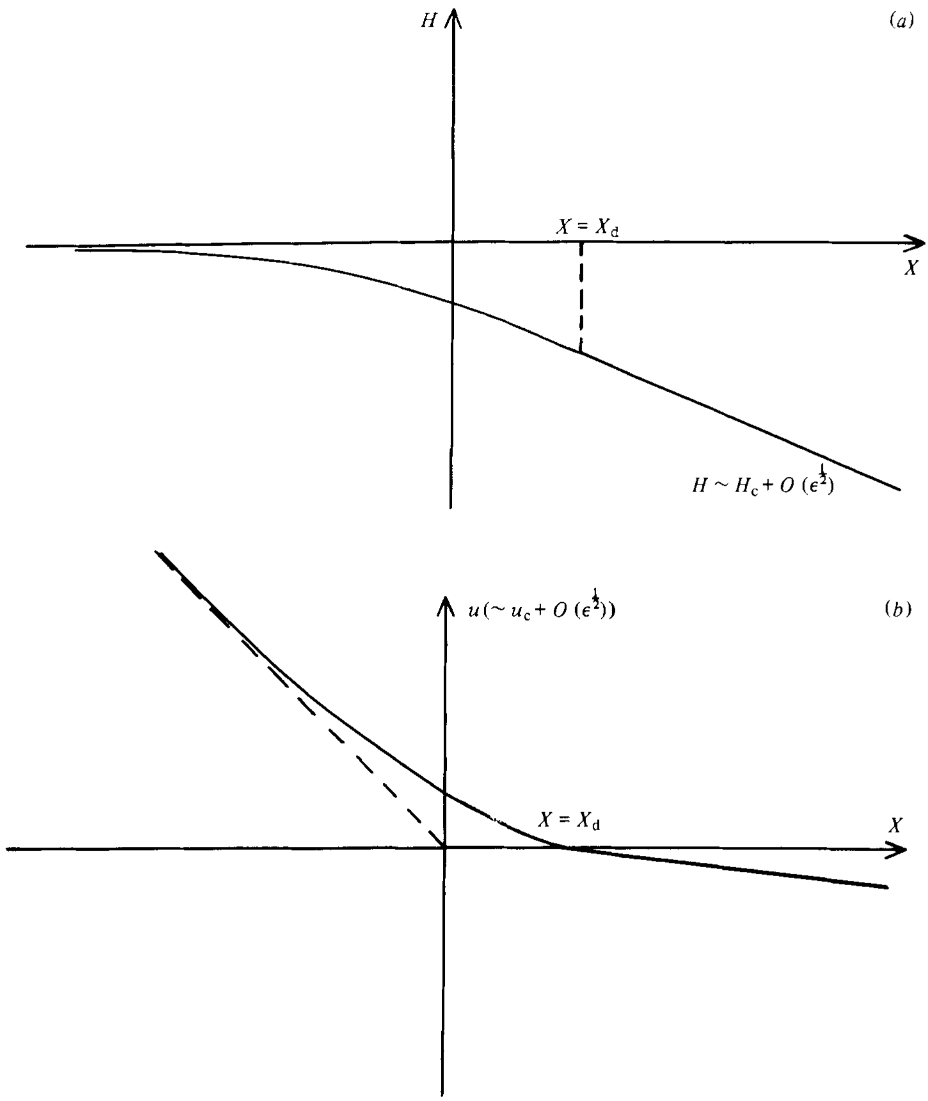

FIGURE 4. Schematic steady-state solutions in the transition region: (*a*) for H; (*b*) for u.

6.3.3. *Continuity considerations at X_d: initial condition and steady state*

We must now complete the formulation by specifying the continuity conditions to be satisfied on X_d, and also the initial condition. Let us first recapitulate what we have found so far. § 6.3.1 describes the matching conditions to be satisfied at $X \to \pm\infty$ ((6.30) and (6.31)). In § 6.3.2 (i) we scale the variables in the subcritical zone $X > X_d$, obtaining the solutions (satisfying the matching condition at $X \to +\infty$) (6.37). These are valid in both steady and time-dependent cases: in the former, we require X_d given by (6.39); in the time-dependent case it is as yet unknown. In § 6.3.2 (ii) we derive the scaled problem for the supercritical zone $X < X_d$: the relevant time-dependent equations are given by (6.44); here $u - u_c = \phi^- \sim 1$. In the steady state, the scaling is slightly different (since $\Phi \equiv 0$), and the solutions are given by (6.46) and (6.47), with A yet to be determined.

The initial condition for the time-dependent problem will be the steady-state solution: let us first complete this solution. We do so by requiring that, on X_d, H, H_X, u and u_X are continuous: continuity of u_{XX} then follows from (6.27). The steady-state solution in $X > X_d$ is given by (6.32), with $\chi^+ \sim \phi^+ \sim O(\epsilon^{\frac{1}{2}})$, X_d given by (6.39). It follows that, on $X = X_d$,

$$\left.\begin{aligned} u = u_c, \quad u_X = -\epsilon^{\frac{1}{2}} a/(H_c + m u_c), \\ H = H_c - \epsilon^{\frac{1}{2}}\left[\frac{\tilde{\mu} m a H_c}{H_c + m u_c}\right], \quad H_X = -\epsilon^{\frac{1}{2}} m a/(H_c + m u_c), \end{aligned}\right\} \tag{6.48}$$

to $O(\epsilon)$. In terms of ϕ^- and χ^-, given by (6.40), we thus require (6.46) and (6.47) to satisfy, on $X = X_d-$,

$$\left.\begin{aligned} \phi^- = 0, \quad \phi_X^- = -\epsilon^{\frac{1}{2}} a/(H_c + m u_c), \\ \chi^- = -\tilde{\mu} m a H_c/(H_c + m u_c), \quad \chi_X^- = -ma/(H_c + m u_c). \end{aligned}\right\} \tag{6.49}$$

We satisfy the condition on χ^- by requiring that

$$A \exp\left[\frac{X_d}{\tilde{\mu} H_c}\right] = \frac{\tilde{\mu} m a H_c}{H_c + m u_c}, \tag{6.50}$$

where X_d is given by (6.39). It is easily checked that the other constraints in (6.49) are then satisfied automatically.

The steady state is thus given by (6.32), (6.38)–(6.40), (6.46), (6.47) and (6.50). A schematic view of the profiles of χ and ϕ is shown in figure 4. Surface slope (μ) diffusion acts to provide a stationary shock structure in the supercritical zone, wherein H changes by $O(\epsilon^{\frac{1}{2}})$ in a length $O(\epsilon^{\frac{1}{2}})$. At this level of approximation, the compressive stress (α) diffusion has a negligible effect.

Now consider the time-dependent problem. The solution in $X > X_d$ is given in § 6.3.2 (i) by (6.32) and (6.37), where X_d is to be determined. Continuity of ϕ^-, ϕ_X^-, χ^- and χ_X^- at X_d then requires that

$$\left.\begin{aligned} \phi^- = 0, \quad \phi_X^- = O(\epsilon^{\frac{1}{2}}), \quad \chi^- = -\frac{ma}{H_c + m u_c}\left[X_d + \frac{\tilde{\mu} H_c^2}{H_c + m u_c}\right] + O(\epsilon^{\frac{1}{2}}), \\ \chi_X^- = \frac{-ma}{H_c + m u_c} + O(\epsilon^{\frac{1}{2}}) \end{aligned}\right\} \tag{6.51}$$

on $X = X_d^-$.

We can now restate the time-dependent problem to be solved as follows. We define

$$\psi = \phi^- - \Phi(\tau) \quad (X < X_d); \tag{6.52}$$

eliminating χ from (6.44) and integrating once yields as the equation for ψ

$$\tilde{\alpha}\psi_{X\tau} + \tilde{\mu} H_c \psi_X - \psi = 0 \quad (X < X_d); \tag{6.53}$$

the matching condition at $X = -\infty$ is

$$\psi \to 0 \quad \text{as} \quad X \to -\infty; \tag{6.54a}$$

the continuity requirements, at leading order, are

$$\psi = -\Phi(\tau), \quad \psi_X = 0 \quad \text{on} \quad X = X_d(\tau); \tag{6.54b}$$

the initial condition (since $\Phi(0) = 0$) is

$$\psi = 0 \quad \text{on} \quad \tau = 0. \tag{6.54c}$$

In terms of ψ, χ^- is given by integrating (6.42) once, using the continuity condition for χ^- in (6.51), to obtain

$$\chi^- = -\exp\left[\frac{X}{\tilde{\mu}H_c}\right]\left[\frac{\tilde{\alpha}}{\tilde{\mu}}\int_X^{X_d(\tau)} \psi_{XX}\exp\left(\frac{-X}{\tilde{\mu}H_c}\right)dX + \left(\frac{ma}{H_c+mu_c}\right)\exp\left(\frac{-X_d(\tau)}{\tilde{\mu}H_c}\right)\left\{X_d(\tau)+\frac{\tilde{\mu}H_c^2}{H_c+mu_c}\right\}\right]. \quad (6.55)$$

We then require the matching and continuity conditions

$$\chi^- \to 0 \quad \text{as} \quad X \to -\infty, \quad (6.56a)$$

$$\chi_X^- = -ma/(H_c+mu_c) \quad \text{on} \quad X = X_d-, \quad (6.56b)$$

and finally the initial conditions, from (6.39), (6.47) and (6.50),

$$X_d(0) = \frac{\tilde{\mu}mH_cu_c}{H_c+mu_c}, \quad (6.57a)$$

$$\chi^-|_{\tau=0} = -\frac{\tilde{\mu}maH_c}{H_c+mu_c}\exp\left\{\frac{X-X_d(0)}{\tilde{\mu}H_c}\right\}. \quad (6.57b)$$

Our problem is thus to solve (6.53)–(6.57) for the unknowns ψ, χ^-, X_d in $\tau \geqslant 0$, $-\infty \leqslant X \leqslant X_d(\tau)$.

6.3.4. *Time-dependent solution; $u > u_c$*

This is not as hard as it looks. Let us suppose $X_d(\tau)$ is known; then we can solve (6.53)–(6.54) for ψ by using Riemann's representation method (e.g. Copson 1975). The details are given in appendix B. The result is

$$\psi(\xi,\eta) = -\Phi(\eta) - \int_{\tau_0(\xi)}^{\eta}\Phi(\tau)\left(\frac{X_d(\tau)-\xi}{\tilde{\alpha}(\eta-\tau)}\right)^{\frac{1}{2}}J_0'\left(2\left(\frac{(X_d(\tau)-\xi)(\eta-\tau)}{\tilde{\alpha}}\right)^{\frac{1}{2}}\right) \times\exp\left[\frac{-\tilde{\mu}H_c(\eta-\tau)}{\tilde{\alpha}}\right]d\tau, \quad (6.58)$$

where $\xi\in(-\infty, X_d(\eta))$ is the space variable and $\eta \geqslant 0$ is the time variable; $\tau_0(\xi)$ is defined by

$$\left.\begin{aligned} \tau_0(\xi) &= 0 \quad (\xi < X_d(0)),\\ X_d[\tau_0(\xi)] &= \xi \quad (\xi > X_d(0)).\end{aligned}\right\} \quad (6.59)$$

Although (6.58) is rather fearsome, we may obtain its asymptotic behaviour as $\xi \to -\infty$, for fixed η; this is given by

$$\psi \sim -\Phi'(0)\,\pi^{\frac{1}{2}}\tilde{\alpha}^{\frac{1}{4}}\frac{\eta^{\frac{1}{4}}}{|\xi|^{\frac{3}{4}}}\cos\left\{2\left(\frac{\eta|\xi|}{\tilde{\alpha}}\right)^{\frac{1}{2}} - \tfrac{3}{4}\pi\right\}\exp\left[-\frac{\tilde{\mu}H_c\eta}{\tilde{\alpha}}\right] \quad (6.60)$$

as $\xi \to -\infty$, where we assume $\Phi(0) = 0$, $\Phi'(0) > 0$ (see appendix B for details).

We can see from (6.60) that the presence of non-zero $\tilde{\alpha}$ makes the decay of ψ (and hence ϕ) oscillatory. For smaller $\tilde{\alpha}$, this oscillation is damped out, as may be seen by putting $\tilde{\alpha} = 0$ in (6.53).

The solution (6.58) satisfies (6.53) and (6.54). It remains to consider (6.55)–(6.57). Integration by parts of (6.55) and consideration of (6.60) shows that (6.56a) is automatically satisfied: it is also easily seen that (6.57b) is true. It only remains to satisfy

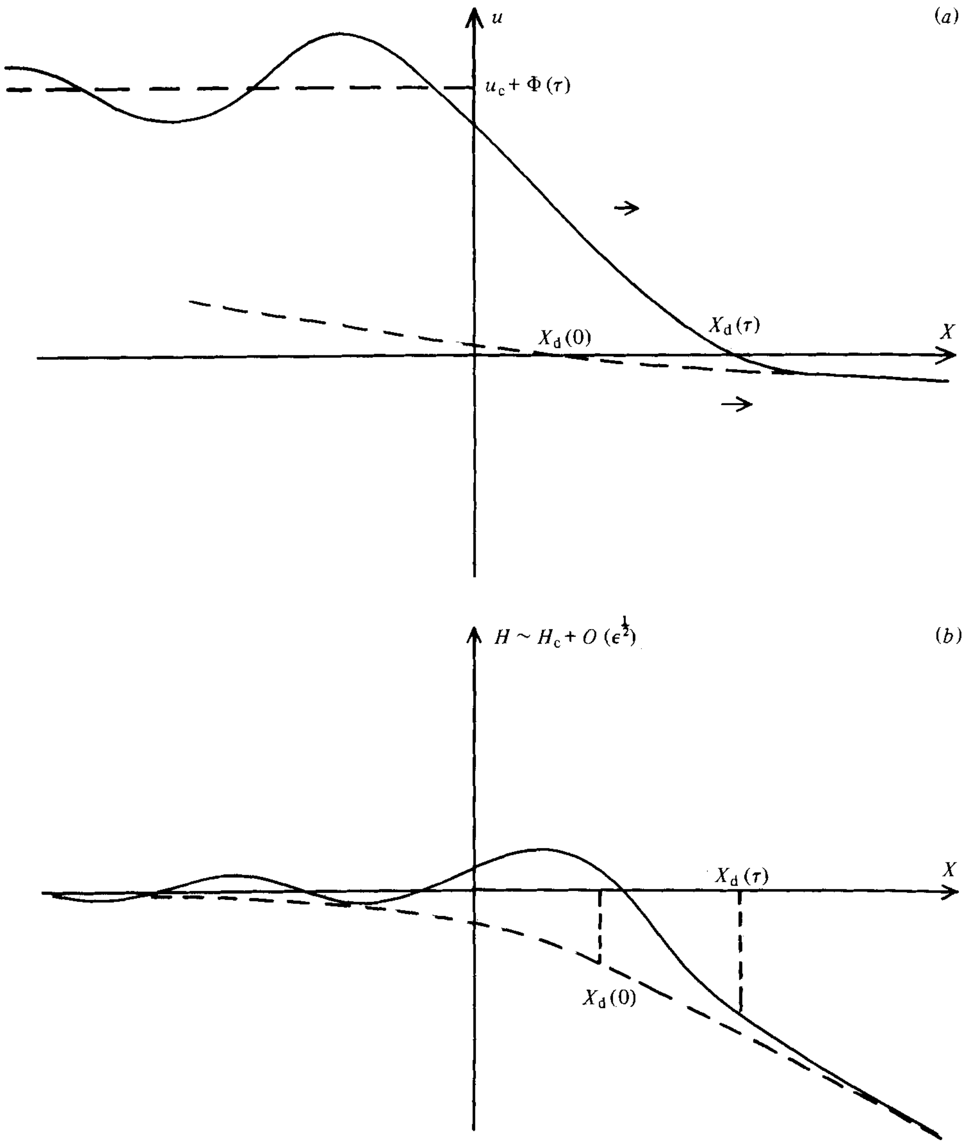

FIGURE 5. Schematic solutions in the transition region: (*a*) for u; (*b*) for H. The steady solutions (figure 4(*a*) and (*b*) for H and u respectively) are shown dotted.

(6.56*b*) and (6.57*a*), and we hope thereby to determine $X_{\mathrm{d}}(\tau)$. From (6.55), we evaluate $\chi_{\bar{X}}$ at X_{d}^- as

$$\chi_{\bar{X}}|_{X_{\mathrm{d}}} = -\frac{1}{\tilde{\mu}H_{\mathrm{c}}}\frac{ma}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\left\{X_{\mathrm{d}}(\tau)+\frac{\tilde{\mu}H_{\mathrm{c}}^2}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\right\}+\frac{\tilde{\alpha}}{\tilde{\mu}}\psi_{XX}|_{X_{\mathrm{d}}}. \quad (6.61)$$

To evaluate ψ_{XX}, we write (6.58) in the form

$$\psi(\xi,\eta) = -\Phi(\eta)-\int_{\tau_0(\xi)}^{\eta}\Phi(\tau)\exp\left[-\frac{\tilde{\mu}H_{\mathrm{c}}(\eta-\tau)}{\tilde{\alpha}}\right] \times\left[-\frac{1}{\tilde{\alpha}}(X_{\mathrm{d}}(\tau)-\xi)+\frac{(\eta-\tau)}{2\tilde{\alpha}^2}(X_{\mathrm{d}}-\xi)^2+O((X_{\mathrm{d}}-\xi)^3)\right]d\tau, \quad (6.62)$$

where we have used the series expansion for J_0, $J_0(z) = 1-\frac{1}{4}z^2+\frac{1}{64}z^4\ldots$. The integrand in (6.62) is an analytic function of ξ, so we may differentiate under the integral sign. Using the definition of $\tau_0(\xi)$, we obtain, after some manipulation,

$$\psi_{\xi\xi}|_{X_{\mathrm{d}}(\eta)} = \frac{\Phi(\eta)}{\tilde{\alpha}X'_{\mathrm{d}}(\eta)}. \tag{6.63}$$

Together with (6.61) and (6.56*b*) this implies

$$-\frac{1}{\tilde{\mu}H_{\mathrm{c}}}\left(\frac{ma}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\right)\left\{X_{\mathrm{d}}(\tau)+\frac{\tilde{\mu}H_{\mathrm{c}}^2}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\right\}+\frac{1}{\tilde{\mu}}\frac{\Phi(\tau)}{X'_{\mathrm{d}}(\tau)} = \frac{-ma}{H_{\mathrm{c}}+mu_{\mathrm{c}}},$$

whence

$$X'_{\mathrm{d}}(\tau) = \frac{H_{\mathrm{c}}\,\Phi(\tau)}{\dfrac{ma}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\left[X_{\mathrm{d}}-\dfrac{\tilde{\mu}mu_{\mathrm{c}}H_{\mathrm{c}}}{H_{\mathrm{c}}+mu_{\mathrm{c}}}\right]}. \tag{6.64}$$

The final condition is (6.57*a*), which is the initial condition for X_{d}. Solving (6.64) and (6.57*a*), we find

$$X_{\mathrm{d}} = \frac{\tilde{\mu}mu_{\mathrm{c}}H_{\mathrm{c}}}{H_{\mathrm{c}}+mu_{\mathrm{c}}}+\left[\frac{2H_{\mathrm{c}}(H_{\mathrm{c}}+mu_{\mathrm{c}})}{ma}\int_0^{\tau}\Phi(\tau)\,d\tau\right]^{\frac{1}{2}}, \tag{6.65}$$

which describes the evolution of the critical point X_{d}. This completes the solution of the transcritical diffusion problem. By comparison, we see that (6.65) is identical to (6.23*a*) if we make the identification

$$x_{\mathrm{d}}^* = X_{\mathrm{d}}-\tilde{\mu}H_{\mathrm{c}}, \tag{6.66}$$

and it follows from the above analysis that the appropriate boundary condition for $x_{\mathrm{d}}^*(0)$ in (6.25) is

$$x_{\mathrm{d}}^*(0) = -\frac{\tilde{\mu}H_{\mathrm{c}}^2}{H_{\mathrm{c}}+mu_{\mathrm{c}}}. \tag{6.67}$$

The forms of the time-dependent solutions for u and H in $-\infty < X < +\infty$ are sketched in figure 5.

7. Summary and conclusions

The precise aim of this paper has been to develop a mathematical description of the two main types of glacial wave motion, and it is hoped that this development is satisfactorily complete. By a mathematical description, we mean a theoretical analysis that explains the pertinent facts, and that is also self-consistent mathematically, being systematically derived, using established approximation methods, from a general continuum model. Since the equations of motion that we consider are complex, it is not surprising that the analysis becomes fairly involved, and we do not offer any apology for this. Nevertheless, the methods presented here are in principle straightforward, and we hope that the results, at least, will be comprehensible to glaciologists (as well as the applied mathematician).

Since the results of the present work deviate in certain respects from those of other authors concerning waves (Nye 1960; Lick 1970; Hutter 1980), let us review the procedure of the paper, and then summarize the relevant physical results.

First, we consider a complete continuum model (Fowler & Larson 1978). This model is then non-dimensionalized using the external inputs (boundary data) in such a way that all variables are of numerical order one, and all parameters are $\lesssim O(1)$. By then judiciously taking asymptotic limits where appropriate, we *derive* from the continuum equations an approximate model, which can be used to study both surface and seasonal waves. In order to simplify the ensuing analysis, we further suppose that basal sliding is the dominant constituent of motion, although we do not believe that this assumption is qualitatively necessary for the wave-propagation mechanisms we discuss.

The dynamics of the resulting, essentially one-dimensional model are crucially dependent on the form of the basal sliding velocity, which prescribes a relation between the basal stress τ_b and the basal velocity u (and possibly other quantities). We take the view that the phenomenon of subglacial cavitation will have a dramatic effect, changing the slope of the stress–velocity curve from an $O(1)$ quantity to an $O(\epsilon)$ quantity ($\epsilon \sim 10^{-2}$) at a critical velocity; we suppose this change occurs discontinuously, i.e. the slope is discontinuous at $u = u_c$. Supposing that τ_b is a function of u only when $u > u_c$ is tantamount to supposing that the cavities are *autonomous*, that is, unconnected to the subglacial drainage system; if we relax this assumption, then τ_b must also be a function of the effective pressure $N =$ (overburden pressure) – (water pressure). However, to assume $\tau_b = f(u)$ is nevertheless physically sensible, and is reasonable here since our aim is to understand the dynamic effects of cavitation, rather than to add to the literature concerned with the sliding law itself.

Our model thus consists of a kinematic wave equation (derived from the mass continuity equation) supplemented by an expression for the flux. We can state the results with reference to the assumptions above, although (in § 4) we have discussed the effect of shearing motion on surface waves. With a velocity u and a depth H (both made dimensionless and $O(1)$), the flux is Hu, the basal sliding law gives $\tau_b = f(u)$, and the model is completed by an expression for τ_b in terms of H and u. This is a slight generalization of Nye's (1952) formula, and includes two dimensionless parameters μ and α; μ multiplies H_x and is therefore a measure of the effect of the surface slope on the stress; α multiplies $(Hu_x)_x$ (in the Newtonian case), and is therefore a measure of the effects of compressive stress. Both these parameters are typically small, estimates of physical relevance being $\mu \sim 10^{-1}$, $\alpha \sim 4 \times 10^{-3}$. In the absence of dynamic wave motions (and hence shocks) it is reasonable to neglect both μ and α to leading order in determining the stress. It then follows that this is given by $\tau_b \simeq H$, and thus the critical velocity u_c of the sliding determines a critical depth $H_c \simeq f(u_c)$ (the precise definition of H_c is slightly different). According to the sliding law, when (at least in a steady state, where we can neglect μ and α) $H \lesssim H_c$, then also $u < u_c$, and so cavitation is absent, and $f'(u) \sim O(1)$; however, when $H \gtrsim H_c$ then $u > u_c$, and $f'(u) \sim O(\epsilon)$; cavitation exists in such regions of greater depth.

It therefore follows that the (steady-state) depth will generally be as shown in figure 3. Since the flux uH through a section is equal to the integral of the accumulation rate ($s(x)$), which will generally be a positive concave downward function (i.e. $d^2s/dx^2 < 0$), the flow will be divided into one or more *supercritical* zones, where $u > u_c$, $H \gtrsim H_c$, i.e. $s(x) \gtrsim H_c u_c$, and *subcritical* zones (at either end) where $u < u_c$, $H \lesssim H_c$, and $s(x) \lesssim H_c u_c$.

If $s < H_c u_c$ throughout, then the whole glacier is subcritical, and time-dependent motions consist of kinematic waves. As these pass downglacier, they evolve nonlinearly and, if $\mu = \alpha = 0$, shocks will ultimately form. In this case μ acts as a diffusive term, smoothing such shocks out monotonically over a length scale $O(\mu)$. In dimensional terms, this is $O(d/\epsilon^*)$ metres, where d is a typical depth (in m) and ϵ^* is the mean bedrock slope. Thus the wave profile becomes sharper for glaciers which flow over steeply sloping bedrock. The relative importance of longitudinal stress in the diffusion region is given by $\alpha/\mu^{1+1/n}$, where n is the exponent in Glen's law; with typical estimates as before, this term is small, and it is consistent to ignore it in the analysis. (This will not be true, though, for steeper bedrocks, when μ becomes smaller.)

Now consider the case shown in figure 3, where there is a supercritical region. Within this region (and neglecting diffusional terms μ and α), the analysis shows that fast, essentially linear waves travel through at a speed $\sim O(1/\epsilon)$ (relative to the surface speed). These waves consist of $O(1)$ variations in velocity, but only $O(\epsilon)$ variations in depth, which, although an important constituent of the motions analytically, will probably be unobservable experimentally. They are not 'compression waves', deriving their properties solely from the assumed form of the supercritical sliding law.

The linearity of supercritical seasonal waves implies that they travel without change of shape, and so a shock-structure analysis corresponding to that for surface waves is irrelevant. However, since supercritical zones are bounded upstream and downstream by 'critical barriers' at which $s(x) \sim H_c u_c$, and beyond which $u < u_c$, the transition behaviour of supercritical waves arriving at a critical barrier is of some interest. The disparity of amplitudes and time scales in the two regions suggests that a *transcritical* shock must then form (in the diffusionless limit) at the critical barrier. The incoming wave has speed $O(1/\epsilon)$, depth amplitude $O(\epsilon)$, velocity amplitude $O(1)$. A shock analysis (with $\mu = \alpha = 0$) shows that a transcritical shock forms across which the velocity jump is $O(1)$, the depth jump is $O(\epsilon^{\frac{1}{2}})$; the shock travels forward a distance $O(\epsilon^{\frac{1}{2}})$ at speed $O(\epsilon^{-\frac{1}{2}})$ as the supercritical wave arrives and is absorbed. When the incoming velocity wave amplitude has decreased to zero, we are left with a depth discontinuity of height $O(\epsilon^{\frac{1}{2}})$, which then proceeds to travel forward as a surface wave into the subcritical zone. Thus the supercritical incoming wave is turned into an outgoing subcritical surface wave by the 'shock-absorbing' critical barrier.

To study the effect of diffusion on this transcritical shock, we consider $\alpha \ll 1, \mu \ll 1$, and motivated by the parametric estimates $\alpha \sim 10^{-3}$, $\epsilon \sim 10^{-2}$, $\mu \sim 10^{-1}$, define $\tilde{\alpha} = \alpha/\epsilon^{\frac{3}{2}} \sim O(1)$, $\tilde{\mu} = \mu/\epsilon^{\frac{1}{2}} \sim O(1)$. It then turns out that the steady-state solution has has a transcritical singular region, as shown in figure 4. A distance $O(\epsilon^{\frac{1}{2}})$ upstream of the critical barrier C, the depth decreases by $O(\epsilon^{\frac{1}{2}})$, although there is no singular region downstream of C in the steady state. As a seasonal wave arrives at C at speed $O(1/\epsilon)$, the amplitude of u upstream jumps rapidly by $O(1)$, causing the transition point C to move forward. The diffusion region is of thickness $O(\epsilon^{\frac{1}{2}})$, and over this u decays *oscillatorily* (if $\tilde{\alpha} \neq 0$) to its incoming amplitude, far from the transcritical region; thus the effect of compressive stress is to introduce an oscillatory space dependence of u (and thus also H) upstream of C.

Downstream of C, little happens; the depth H is virtually unaffected, and there is a small singular region of width $O(\epsilon^{\frac{3}{4}})$ over which u changes by $O(\sqrt{\epsilon})$. These conclusions are represented pictorially in figure 5.

Finally, let us note that there are many facets of the real flow which we have ignored: for example the shearing motion (in § 6), non-uniformity of the transverse valley profile (Nye's shape factor), the effect of temperature variation, the variation of the bedrock

$h \neq 0$, flow round corners, incorporation of effective pressure and discussion of subglacial drainage in the sliding law. The reasons for such omissions are various: irrelevance to the topic at hand, resulting simplifications, and so on. Let us, however, re-emphasize the scope and nature of an analysis such as that presented here. Given a set of observations, we seek in a mathematical framework to provide an (at least) qualitative explanation of their behaviour, and to give (rough) quantitative descriptions as well. In so doing, we retain *only* that which is considered relevant to the specific observations. Such simplified models offer primarily the ability to understand complex phenomena and also serve in elucidating and predicting the actual behaviour observed, both on the computer and in the field.

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Appendix A. Scaling the glacier model when basal slip is dominant

In this appendix we derive the appropriate scaling of the model equations of glacier flow when the velocity due to shearing is asymptotically smaller than that due to basal slip. The development parallels that of Fowler & Larson (1978), to which the reader is referred.

The longitudinal velocity is written as

$$u = U[u_b + \beta u_1]; \tag{A 1}$$

here U is a velocity scale, u_b and u_1 are dimensionless velocities due to slip and shearing respectively. We suppose $u_b \sim 1$, $u_1 \sim 1$, $\beta \ll 1$. If d (to be determined) and l are relevant depth and length scales, then the two-dimensional continuity equation implies that the vertical velocity v may be written as

$$v = \delta U[-u_b' y + \beta v_1], \tag{A 2}$$

where $\delta = d/l$ is the aspect ratio, and U is found by prescribing that δU is of the order of magnitude of the surface accumulation rate. In (A 2), y is the dimensionless vertical co-ordinate: u_1 and v_1 satisfy

$$u_{1x} + v_{1y} = 0. \tag{A 3}$$

Glen's flow law

$$e_{ij} = A\tau^{n-1}\tau_{ij}, \quad e = A\tau^n \tag{A 4}$$

may be written, with $\tau_{12} = \tau_{21} = \tau_2$, $\tau_{11} = -\tau_{22} = \tau_1$, as

$$\tau_{ij} = A^{-1/n} e^{1/n-1} e_{ij}, \tag{A 5}$$

whence

$$\left.\begin{aligned} \tau_1 &= A^{-1/n} u_x e^{1/n-1}, \\ \tau_2 &= \tfrac{1}{2} A^{-1/n}(u_y + v_x)\, e^{1/n-1}, \end{aligned}\right\} \tag{A 6}$$

where

$$e = \tfrac{1}{2}[4u_x^2 + (u_y + v_x)^2]^{\frac{1}{2}}. \tag{A 7}$$

Using (A 1) and (A 2), (A 7) is

$$e = \frac{1}{2}\left[\frac{4U^2}{l^2}\{u_\mathrm{b}' + \beta u_{1x}\}^2 + \frac{U^2}{d^2}\{\beta u_{1y} - \delta^2 y u_\mathrm{b}'' + \delta^2 \beta v_{1x}\}^2\right]^{\frac{1}{2}}$$

$$= \frac{U}{2d}[4\delta^2\{u_\mathrm{b}' + \beta u_{1x}\}^2 + \{\beta u_{1y} - \delta^2 y u_\mathrm{b}'' + \delta^2 \beta v_{1x}\}^2]^{\frac{1}{2}}$$

$$= \frac{\beta U}{2d}\bar{e}, \tag{A 8}$$

where

$$\bar{e} = [4\nu^2\{u_\mathrm{b}' + \beta u_{1x}\}^2 + \{u_{1y} - \delta\nu y u_\mathrm{b}'' + \delta^2 v_{1x}\}^2]^{\frac{1}{2}}, \tag{A 9}$$

$$\delta = \nu\beta, \quad \nu \lesssim 1. \tag{A 10}$$

(A 6) gives

$$\tau_1 = A^{-1/n}\left(\frac{\beta U}{2d}\right)^{1/n-1} \bar{e}^{1/n-1}\frac{U\delta}{d}[u_\mathrm{b}' + \beta u_{1x}], \tag{A 11}$$

$$\tau_2 = \tfrac{1}{2}A^{-1/n}\left(\frac{\beta U}{2d}\right)^{1/n-1} \bar{e}^{1/n-1}\frac{\beta U}{d}[u_{1y} - \delta\nu y u_\mathrm{b}'' + \delta^2 v_{1x}]. \tag{A 12}$$

We now define the stress scale

$$[\tau] = \left(\frac{\beta U}{2Ad}\right)^{1/n}, \tag{A 13}$$

and write

$$\tau_2 = [\tau]\bar{\tau}_2, \quad \tau_1 = \nu[\tau]\bar{\tau}_1, \tag{A 14}$$

so that (A 11) and (A 12) are

$$\left.\begin{aligned} \bar{\tau}_2 &= \bar{e}^{1/n-1}[u_{1y} - \delta\nu u_\mathrm{b}'' y + \delta^2 v_{1x}], \\ \bar{\tau}_1 &= 2\bar{e}^{1/n-1}[u_\mathrm{b}' + \beta u_{1x}]. \end{aligned}\right. \tag{A 15}$$

We scale the pressure minus its hydrostatic component in the same way as τ_1; thus we put

$$p = p_\mathrm{A} + \rho g' d(\eta - y) + \nu[\tau]\bar{p}. \tag{A 16}$$

Substitution of (A 15) and (A 16) into the momentum equations

$$\left.\begin{aligned} \rho[u_t + uu_x + vu_y] &= -p_x + \tau_{1x} + \tau_{2y} + \rho g'\epsilon^*, \\ \rho[v_t + uv_x + vv_y] &= -p_y + \tau_{2x} - \tau_{1y} - \rho g', \end{aligned}\right\} \tag{A 17}$$

now gives (3.5) directly; the surface boundary conditions are found likewise. This is effected by choosing the same balance of stress and gravity, that is $\tau_{2y} \sim \rho g'\epsilon^*$, whence we specify

$$[\tau] = \rho g'\epsilon^* d. \tag{A 18}$$

The two relations (A 13) and (A 18) serve to specify d completely; the only difference between the case $\beta \sim 1$ and $\beta \ll 1$ is the factor β in (A 13). Since from (A 13) and (A 18) we find

$$(\rho g'\epsilon^*)^n d^n = \frac{\beta U}{2Ad} = \frac{\beta l V}{2Ad^2}, \quad V = \delta U, \tag{A 19}$$

there follows

$$d = \left[\frac{\beta l V}{2A(\rho g'\epsilon^*)^n}\right]^{1/(n+2)}. \tag{A 20}$$

This gives d in terms of physically prescribed parameters (V is prescribed and not U). The factor $\beta^{1/(n+2)}$ is of little numerical significance; for example, if $\beta \sim \frac{1}{30}$, $n \simeq 3$, then $\beta^{1/(n+2)} \sim \frac{1}{2}$.

Appendix B

Here we derive the solution of (6.53), (6.54) and its asymptotic form as $X \to -\infty$. The problem is

$$\left.\begin{aligned} &\tilde{\alpha}\psi_{X\tau}+\tilde{\mu}H_c\psi_X-\psi = 0 \quad (X < X_d); \\ &\psi = 0 \quad \text{on} \quad \tau = 0 \quad \text{and} \quad X = -\infty; \\ &\psi = -\Phi(\tau), \quad \psi_X = 0 \quad \text{on} \quad X = X_d(\tau). \end{aligned}\right\} \tag{B 1}$$

We use Riemann's representation method (Copson 1975). The line $\tau = 0$ is a characteristic, but the data $\psi = 0$ on it is consistent with the equation, and a solution may be found in the usual way. Specifically, the value of ψ at a point (ξ, η) in (X, τ)-space ($\xi < X_d(\tau)$, $\eta > \tau$) may be found in terms of a Riemann–Green function $R(X, \tau; \xi, \eta)$ which satisfies

$$\begin{aligned} &\tilde{\alpha}R_{X\tau}-\tilde{\mu}H_c R_X - R = 0 \quad (\xi \leqslant X \leqslant X_d(\tau), \quad 0 \leqslant \tau \leqslant \eta), \\ &R_X = 0 \quad \text{on} \quad \tau = \eta, \\ &\tilde{\alpha}R_\tau - \tilde{\mu}H_c R = 0 \quad \text{on} \quad X = \xi, \\ &R(\xi, \eta; \xi, \eta) = 1. \end{aligned} \tag{B 2}$$

With the boundary conditions on X_d in (B 1), ψ is given by

$$\psi(\xi, \eta) = -\Phi(\eta) + \int_C \Phi(\tau)\left[R_\tau - \frac{\tilde{\mu}H_c}{\tilde{\alpha}}R\right] d\tau, \tag{B 3}$$

where C is the portion of the curve $X = X_d(\tau)$ between $X = \max\{\xi, X_d(0)\}$ and $X = X_d(\eta)$. (We assume $X_d' > 0$, as subsequently verified in the main text.) (B 2) is easily solved to find

$$R[X, \tau; \xi, \eta] = \exp\left[-\frac{\tilde{\mu}H_c}{\tilde{\alpha}}(\eta-\tau)\right] J_0\left[2\left(\frac{(X-\xi)(\eta-\tau)}{\tilde{\alpha}}\right)^{\frac{1}{2}}\right], \tag{B 4}$$

whence the explicit form for $\psi(\xi, \eta)$ is

$$\psi(\xi, \eta) = -\Phi(\eta) - \int_{\tau_0(\xi)}^{\eta} \Phi(\tau)\left(\frac{X_d(\tau)-\xi}{\tilde{\alpha}(\eta-\tau)}\right)^{\frac{1}{2}} J_0'\left[2\left(\frac{(X_d(\tau)-\xi)(\eta-\tau)}{\tilde{\alpha}}\right)^{\frac{1}{2}}\right] \times \exp\left[-\frac{\tilde{\mu}H_c(\eta-\tau)}{\tilde{\alpha}}\right] d\tau, \tag{B 5}$$

where $\tau_0(\xi)$ is given by

$$\left.\begin{aligned} \tau_0(\xi) &= 0 \quad (\xi < X_d(0)), \\ X_d[\tau_0(\xi)] &= \xi \quad (\xi > X_d(0)). \end{aligned}\right\} \tag{B 6}$$

It is easy to see that (B 5) satisfies $\psi = 0$ on $\eta = 0$. To check the condition at $X \to -\infty$, we refer to the asymptotic analysis below.

To consider the asymptotic behaviour of ψ as $\xi \to -\infty$, define

$$z = 2\left[\frac{(X_d(\tau)+|\xi|)(\eta-\tau)}{\tilde{\alpha}}\right]^{\frac{1}{2}}, \tag{B 7}$$

$$\zeta = \frac{\tilde{\alpha}z^2}{4|\xi|} = (\eta-\tau)\left(1+\frac{X_d(\tau)}{|\xi|}\right). \tag{B 8}$$

Assume $\Phi(\tau)$ and $X_{\mathrm{d}}(\tau)$ are analytic functions of τ. (If Φ is, then X_{d} is also, *a posteriori* from (6.65), if $\Phi' > 0$.) Then $\zeta(\tau)$ is analytic, and so also $\tau = \tau(\zeta)$. Note that

$$\left.\begin{aligned} \eta-\tau &= \zeta\left(1+\frac{X_{\mathrm{d}}(\tau)}{|\xi|}\right)^{-1} = \zeta+O\left(\frac{1}{|\xi|}\right), \\ -\frac{d\tau}{d\zeta} &= 1+O\left(\frac{1}{|\xi|}\right). \end{aligned}\right\} \tag{B 9}$$

We further define

$$\left.\begin{aligned} \zeta^* &= \eta\left[1+\frac{X_{\mathrm{d}}(0)}{|\xi|}\right] = \eta+O\left(\frac{1}{|\xi|}\right), \\ z^* &= 2\left(\frac{|\xi|\,\zeta^*}{\tilde{\alpha}}\right)^{\frac{1}{2}} = 2\left(\frac{|\xi|\,\eta}{\tilde{\alpha}}\right)^{\frac{1}{2}}+O\left(\frac{1}{|\xi|^{\frac{1}{2}}}\right); \end{aligned}\right\} \tag{B 10}$$

then (B 5) may be written in the form

$$\psi(\xi,\eta) = -\Phi(\eta)-\int_0^{z^*} G(\zeta;\,|\xi|)\,J_0'(z)\,dz, \tag{B 11}$$

where

$$G(\zeta;\,|\xi|) = \Phi(\tau(\zeta))\left[1+\frac{X_{\mathrm{d}}(T)}{|\xi|}\right]\left(-\frac{d\tau}{d\zeta}\right)\exp\left[-\frac{\tilde{\mu}H_{\mathrm{c}}}{\tilde{\alpha}}\zeta\left(1+\frac{X_{\mathrm{d}}(\tau)}{|\xi|}\right)^{-1}\right] \tag{B 12}$$

is an analytic function of ζ. Integration by parts of (B 11) then yields

$$\begin{aligned} \psi(\xi,\eta) = &-\Phi(\eta)+G(0;\,|\xi|)-J_0(z^*)\,G(\zeta^*;\,|\xi|) \\ &+\frac{\tilde{\alpha}}{2|\xi|}z^*J_1(z^*)\,G_\zeta(\zeta^*;\,|\xi|)-\left(\frac{\tilde{\alpha}}{2|\xi|}\right)^2 z^{*2}J_2(z^*)\,G_{\zeta\zeta}(\zeta^*;\,|\xi|) \\ &+\left(\frac{\tilde{\alpha}}{2|\xi|}\right)^3\int_0^{z^*} G_{\zeta\zeta\zeta}(\zeta^*;\,|\xi|)\,z^3J_2(z)\,dz, \end{aligned} \tag{B 13}$$

where we make use of the recursive formula $d(z^\nu J_\nu(z))/dz = z^\nu J_{\nu-1}(z)$ (Carrier, Krook & Pearson 1966). Further terms may be obtained easily, and it is obvious, since G is analytic in ζ and $1/|\xi|$ (by inspection), and since $|J_\nu(z^*)| \sim z^{*-\frac{1}{2}}$ for all $\nu > 0$ as $z^* \to \infty$, that the series (B 13) is asymptotic. We find, using (B 9), that

$$\begin{aligned} G(0;\,|\xi|) &= \Phi(\eta), \\ G(\zeta^*;\,|\xi|) &= O\left(\frac{1}{|\xi|}\right), \\ G_\zeta(\zeta^*;\,|\xi|) &= -\Phi'(0)\exp\left[-\frac{\tilde{\mu}H_{\mathrm{c}}\zeta^*}{\tilde{\alpha}}\right]+O\left(\frac{1}{|\xi|}\right), \end{aligned} \tag{B 14}$$

if $\Phi(0) = 0$. In this case, the leading-order asymptotic expansion for ψ as $\xi \to -\infty$ is

$$\psi \sim -\Phi'(0)\,\pi^{\frac{1}{2}}\tilde{\alpha}^{\frac{3}{4}}\frac{\eta^{\frac{1}{4}}}{|\xi|^{\frac{3}{4}}}\cos\left\{2\left(\frac{\eta\,|\xi|}{\tilde{\alpha}}\right)^{\frac{1}{2}}-\tfrac{3}{4}\pi\right\}\exp\left[-\frac{\tilde{\mu}H_{\mathrm{c}}\eta}{\tilde{\alpha}}\right], \tag{B 15}$$

valid for fixed η as $|\xi| \to \infty$.

REFERENCES

AHARON, P., CHAPPEL, J. & COMPSTON, W. 1980 Stable isotope and sea level data from New Guinea supports Antarctic ice surge theory of ice ages. *Nature* **283**, 649–651.

BATCHELOR, G. K. 1967 *An Introduction to Fluid Mechanics*. Cambridge University Press.

BINDSCHADLER, R. 1979 Basal sliding and bed separation: is there a connection? *J. Glaciol.* **23**, 407–408.

BIRD, R. B. 1976 Useful non-Newtonian models. *Ann. Rev. Fluid Mech.* **8**, 13–34.

BLÜMCKE, A. & FINSTERWALDER, S. 1905 Zeitliche Änderungen in der Geschwindigkeit der Gletscherbewegung. *Sitz. k. Bayerischen Akad. Wiss., Math.-phys. Klasse* **35**, 109–131.

BOWEN, D. Q. 1980 Antarctic ice surges and theories of glaciation. *Nature* **283**, 619–620.

BUDD, W. F. 1975 A first simple model for periodically self surging glaciers. *J. Glaciol.* **14**, 3–21.

BUDD, W. F. & RADOK, U. 1971 Glaciers and other large ice masses. *Rep. Prog. Phys.* **34**, 1–70.

CAMPBELL, W. J. & RASMUSSEN, L. A. 1969 Three-dimensional surges and recoveries in a numerical glacier model. *Can. J. Earth Sci.* **6**, 979–986.

CARRIER, G. F., KROOK, M. & PEARSON, C. E. 1966 *Functions of a Complex Variable*. McGraw-Hill.

CARY, P. W., CLARKE, G. K. C. & PELTIER, W. R. 1979 A creep instability analysis of the Antarctic and Greenland ice sheets. *Can. J. Earth Sci.* **16**, 182–188.

CLARKE, G. K. C., NITSAN, U. & PATERSON, W. S. B. 1977 Strain heating and creep instability in glaciers and ice sheets. *Rev. Geophys. Space Phys.* **15**, 235–247.

COLBECK, S. C. & EVANS, R. J. 1973 A flow law for temperate glaciers. *J. Glaciol.* **12**, 71–86.

COLE, J. D. 1968 *Perturbation Methods in Applied Mathematics*. Blaisdell.

COLLINS, I. F. 1968 On the use of equilibrium equations and flow law in relating surface and bed topography of glaciers and ice sheets. *J. Glaciol.* **7**, 199–204.

COPSON, E. T. 1975 *Partial Differential Equations*. Cambridge University Press.

DEELEY, R. M. & PARR, P. H. 1914 On the Hintereis glacier. *Phil. Mag.* **27** (6), 153–176.

DORSEY, N. E. 1940 *Properties of Ordinary Water-Substance*. Reinhold.

DRAKE, L. D. & SHREVE, R. L. 1973 Pressure melting and regelation of ice by round wires. *Proc. R. Soc. Lond.* A **332**, 51–83.

FINSTERWALDER, S. 1907 Die Theorie der Gletscherschwankungen. *Z. Gletscherkunde* **2**, 81–103.

FOWLER, A. C. 1979*a* The use of a rational model in the mathematical analysis of a polythermal glacier. *J. Glaciol.* **24**, 443–456.

FOWLER, A. C. 1979*b* A mathematical approach to the theory of glacier sliding. *J. Glaciol.* **23**, 131–141.

FOWLER, A. C. 1980 The existence of multiple steady states in the flow of large ice masses. *J. Glaciol.* **25**, 183–184.

FOWLER, A. C. 1981 A theoretical treatment of the sliding of glaciers in the absence of cavitation. *Phil. Trans. R. Soc. Lond.* A **298**, 637–684.

FOWLER, A. C. & LARSON, D. A. 1978 On the flow of polythermal glaciers I. Model and preliminary analysis. *Proc. R. Soc. Lond.* A **363**, 217–242.

FOWLER, A. C. & LARSON, D. A. 1980*a* The uniqueness of steady state flows of glaciers and ice sheets. *Geophys. J. R. Astr. Soc.* **63**, 333–345.

FOWLER, A. C. & LARSON, D. A. 1980*b* On the flow of polythermal glaciers II. Surface wave analysis. *Proc. R. Soc. Lond.* A **370**, 155–171.

FOWLER, A. C. & LARSON, D. A. 1980*c* Thermal stability properties of a model of glacier flow. *Geophys. J. R. Astr. Soc.* **63**, 347–359.

GAVALAS, G. R. 1968 *Nonlinear Differential Equations of Chemically Reacting Systems*. Springer.

GLEN, J. W. 1955 The creep of polycrystalline ice. *Proc. R. Soc. Lond.* A **228**, 519–538.

GRUNTFEST, I. J. 1963 Thermal feedback in liquid flow: plane shear at constant stress. *Trans. Soc. Rheol.* **7**, 195–207.

HODGE, S. M. 1974 Variations in the sliding of a temperate glacier. *J. Glaciol.* **13**, 349–369.

HOLLIN, J. T. 1980 Climate and sea-level in isotope stage 5: an East Antarctic ice surge at $\sim$ 95000 BP? *Nature* **283**, 629–633.

HUTTER, K. 1980 Time dependent surface elevation of an ice slope. *J. Glaciol.* **25**, 247–266.

HUTTER, K. 1981 The effect of longitudinal strain on the shear stress of an ice sheet: in defense of using stretched coordinates. *J. Glaciol.* **27**, 39–56.

KAMB, W. B. 1970 Sliding motion of glaciers: theory and observation. *Rev. Geophys. Space Phys.* **8**, 673–728.

KAMB, W. B. & LACHAPELLE, E. 1964 Direct observations of the mechanism of glacier sliding over bedrock. *J. Glaciol.* **5**, 159–172.

LICK, W. 1970 The propagation of disturbances on glaciers. *J. Geophys. Res.* **75**, 2189–2197.

LLIBOUTRY, L. A. 1965 *Traité de Glaciologie*, t. 2. Masson.

LLIBOUTRY, L. A. 1968 General theory of subglacial cavitation and sliding of temperate glaciers. *J. Glaciol.* **7**, 21–58.

LLIBOUTRY, L. A. 1971 The glacier theory. In *Advances in Hydroscience*, vol. 7 (ed. Ven te Chow), pp. 81–167.

LLIBOUTRY, L. A. 1976 Physical processes in temperate glaciers. *J. Glaciol.* **16**, 151–158.

LLIBOUTRY, L. A. 1978 Glissement d'un glacier sur un plan parsemé d'obstacles hémisphériques. *Ann. Géophys.* **34**, 147–162.

LLIBOUTRY, L. A. 1979 Local friction laws for glaciers: a critical review and new openings. *J. Glaciol.* **23**, 67–95.

LLIBOUTRY, L. A. & RAYNEAUD, L. 1981 Global dynamics of a temperate valley glacier, Mer de Glace, and past velocities deduced from Forbes bands. *J. Glaciol.* **27**, 207–226.

MEIER, M. F. 1979 Variations in time and space of the velocity of lower Columbia Glacier, Alaska. *J. Glaciol.* **23**, 408.

MEIER, M. F. & POST, A. S. 1969 What are glacier surges? *Can. J. Earth Sci.* **6**, 807–817.

MORLAND, L. W. 1976 Glacier sliding down an inclined wavy bed. *J. Glaciol.* **17**, 447–462.

MORRIS, E. M. 1976 An experimental study of the motion of ice past obstacles by the process of regelation. *J. Glaciol.* **17**, 79–98.

MORRIS, E. M. 1979 The flow of ice, treated as a newtonian viscous liquid, around a cylindrical obstacle near the bed of a glacier. *J. Glaciol.* **23**, 117–129.

MURRAY, J. D. 1970 Perturbation effects on the decay of discontinuous solutions of nonlinear first order wave equations. *SIAM J. Appl. Math.* **19**, 273–298.

NYE, J. F. 1952 A method of calculating the thickness of the ice sheets. *Nature* **169**, 529–530.

NYE, J. F. 1960 The response of glaciers and ice-sheets to seasonal and climatic changes. *Proc. R. Soc. Lond.* A **256**, 559–584.

NYE, J. F. 1963 On the theory of the advance and retreat of glaciers. *Geophys. J.R. Astr. Soc.* **7**, 431–456.

NYE, J. F. 1967 Theory of regelation. *Phil. Mag.* **16** (8), 1249–1266.

NYE, J. F. 1969 A calculation of the sliding of ice over a wavy surface using a newtonian viscous approximation. *Proc. R. Soc. Lond.* A **311**, 445–467.

NYE, J. F. 1970 Glacier sliding without cavitation in a linear viscous approximation. *Proc. R. Soc. Lond.* A **315**, 381–403.

NYE, J. F. 1973 The motion of ice past obstacles. In *Physics and Chemistry of Ice* (ed. E. Whalley, S. J. Jones & L. W. Gold), pp. 387–394. Ottawa: Royal Society of Canada.

PATERSON, W. S. B. 1969 *The Physics of Glaciers*. Pergamon.

PEARSON, J. R. A. 1977 Variable-viscosity flows in channels with high heat generation. *J. Fluid Mech.* **83**, 191–206.

PEARSON, J. R. A. 1978 Polymer flows dominated by high heat generation and low heat transfer. *Polymer Engng Sci.* **18**, 222–229.

RICHARDSON, S. 1973 On the no-slip boundary condition. *J. Fluid Mech.* **59**, 707–719.

ROBIN, G. DE Q. 1955 Ice movement and temperature distribution in glaciers and ice sheets. *J. Glaciol.* **2**, 523–532.

ROBIN, G. DE Q. 1967 Surface topography of ice sheets. *Nature* **215**, 1029–1032.

ROBIN, G. DE Q. 1969 Initiation of glacier surges. *Can. J. Earth Sci.* **6**, 919–928.

ROBIN, G. DE Q. & WEERTMAN, J. 1973 Cyclic surging of glaciers. *J. Glaciol.* **12**, 3–18.

SMIRNOVA, G. N. 1963 Linear parabolic equations which degenerate on the boundary of the region (in Russian). *Sibirsk Mat. Ž.* **4**, 343–358.

STOCKER, R. L. & ASHBY, M. F. 1973 On the rheology of the upper mantle. *Rev. Geophys. Space Phys.* **11**, 391–426.

THEAKSTONE, W. H. 1979 Observations within cavities at the bed of the glacier Østerdalsisen, Norway. *J. Glaciol.* **23**, 273–281.

Turcotte, D. L. & Oxburgh, E. R. 1972 Mantle convection and the new global tectonics. *Ann. Rev. Fluid Mech.* **4**, 33–68.

Van Dyke, M. D. 1975 *Perturbation Methods in Fluid Mechanics.* Parabolic.

Vivian, R. A. & Bocquet, G. 1973 Subglacial cavitation phenomena under the Glacier d'Argentière, Mont Blanc, France. *J. Glaciol.* **12**, 439–451.

Weertman, J. 1957 On the sliding of glaciers. *J. Glaciol.* **3**, 33–38.

Weertman, J. 1958 Traveling waves on glaciers. In *Proc. IUGG Symp. Chamonix, France*, pp. 162–168. Publication no. 47 de l'Association Internationale d'Hydrologie Scientifique.

Weertman, J. 1964 The theory of glacier sliding. *J. Glaciol.* **5**, 287–303.

Weertman, J. 1977 Penetration depth of closely spaced water-free crevasses. *J. Glaciol.* **18**, 37–46.

Weertman, J. 1979 The unsolved general glacier sliding problem. *J. Glaciol.* **23**, 97–115.

Whitham, G. B. 1974 *Linear and Nonlinear Waves.* Wiley-Interscience.

Yuen, D. A. & Schubert, G. 1977 Asthenospheric shear flow: thermally stable or unstable? *Geophys. Res. Lett.* **4**, 503–506.

Yuen, D. A. & Schubert, G. 1979 The role of shear heating in the dynamics of large ice masses. *J. Glaciol.* **24**, 195–212.